

Chapman & Hall/CRC FINANCIAL MATHEMATICS SERIES

# **The Financial Mathematics of Market Liquidity**

**From Optimal Execution to Market Making**

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