Special Functions and Orthogonal Polynomials

RICHARD BEALS
Yale University

RODERICK WONG
City University of Hong Kong
Contents

Preface

1 Orientation
1.1 Power series solutions 2
1.2 The gamma and beta functions 5
1.3 Three questions 6
1.4 Other special functions 10
1.5 Exercises 11
1.6 Remarks 13

2 Gamma, beta, zeta
2.1 The gamma and beta functions 16
2.2 Euler's product and reflection formulas 19
2.3 Formulas of Legendre and Gauss 22
2.4 Two characterizations of the gamma function 24
2.5 Asymptotics of the gamma function 26
2.6 The psi function and the incomplete gamma function 30
2.7 The Selberg integral 32
2.8 The zeta function 35
2.9 Exercises 38
2.10 Remarks 45

3 Second-order differential equations
3.1 Transformations and symmetry 47
3.2 Existence and uniqueness 49
3.3 Wronskians, Green's functions, and comparison 52
3.4 Polynomials as eigenfunctions 55
3.5 Maxima, minima, and estimates 60
3.6 Some equations of mathematical physics 62
3.7 Equations and transformations 66
3.8 Exercises 68
3.9 Remarks 71

4 Orthogonal polynomials on an interval 73
4.1 Weight functions and orthogonality 74
4.2 Stieltjes transform and Padé approximants 78
4.3 Padé approximants and continued fractions 81
4.4 Generalization: measures 84
4.5 Favard’s theorem and the moment problem 86
4.6 Asymptotic distribution of zeros 89
4.7 Exercises 90
4.8 Remarks 93

5 The classical orthogonal polynomials 94
5.1 Classical polynomials: general properties, I 94
5.2 Classical polynomials: general properties, II 98
5.3 Hermite polynomials 102
5.4 Laguerre polynomials 108
5.5 Jacobi polynomials 111
5.6 Legendre and Chebyshev polynomials 115
5.7 Distribution of zeros and electrostatics 120
5.8 Expansion theorems 124
5.9 Functions of the second kind 130
5.10 Exercises 133
5.11 Remarks 137

6 Semi-classical orthogonal polynomials 140
6.1 Discrete weights and difference operators 141
6.2 The discrete Rodrigues formula 146
6.3 Charlier polynomials 149
6.4 Krawtchouk polynomials 152
6.5 Meixner polynomials 155
6.6 Chebyshev–Hahn polynomials 158
6.7 Neo-classical polynomials 162
6.8 Exercises 168
6.9 Remarks 170

7 Asymptotics of orthogonal polynomials: two methods 172
7.1 Approximation away from the real line 173
7.2 Asymptotics by matching 175
7.3 The Riemann–Hilbert formulation 178
7.4 The Riemann–Hilbert problem in the Hermite case, I 179
7.5 The Riemann–Hilbert problem in the Hermite case, II 185
7.6 Hermite asymptotics 192
8 Confluent hypergeometric functions
  8.1 Kummer functions
  8.2 Kummer functions of the second kind
  8.3 Solutions when c is an integer
  8.4 Special cases
  8.5 Contiguous functions
  8.6 Parabolic cylinder functions
  8.7 Whittaker functions
  8.8 Exercises
  8.9 Remarks

9 Cylinder functions
  9.1 Bessel functions
  9.2 Zeros of real cylinder functions
  9.3 Integral representations
  9.4 Hankel functions
  9.5 Modified Bessel functions
  9.6 Addition theorems
  9.7 Fourier transform and Hankel transform
  9.8 Integrals of Bessel functions
  9.9 Airy functions
  9.10 Exercises
  9.11 Remarks

10 Hypergeometric functions
  10.1 Solutions of the hypergeometric equation
  10.2 Linear relations of solutions
  10.3 Solutions when c is an integer
  10.4 Contiguous functions
  10.5 Quadratic transformations
  10.6 Integral transformations and special values
  10.7 Exercises
  10.8 Remarks

11 Spherical functions
  11.1 Harmonic polynomials and surface harmonics
  11.2 Legendre functions
  11.3 Relations among the Legendre functions
  11.4 Series expansions and asymptotics
  11.5 Associated Legendre functions
  11.6 Relations among associated functions
11.7 Exercises 301
11.8 Remarks 303

12 Generalized hypergeometric functions; G-functions 305
12.1 Generalized hypergeometric series 305
12.2 The generalized hypergeometric equation 308
12.3 Meijer G-functions 312
12.4 Choices of contour of integration 319
12.5 Expansions and asymptotics 322
12.6 The Mellin transform and G-functions 325
12.7 Exercises 326
12.8 Remarks 328

13 Asymptotics 330
13.1 Hermite and parabolic cylinder functions 331
13.2 Confluent hypergeometric functions 333
13.3 Hypergeometric functions and Jacobi polynomials 338
13.4 Legendre functions 340
13.5 Steepest descents and stationary phase 342
13.6 Exercises 345
13.7 Remarks 357

14 Elliptic functions 358
14.1 Integration 359
14.2 Elliptic integrals 361
14.3 Jacobi elliptic functions 366
14.4 Theta functions 371
14.5 Jacobi theta functions and integration 375
14.6 Weierstrass elliptic functions 380
14.7 Exercises 383
14.8 Remarks 388

15 Painlevé transcendentals 390
15.1 The Painlevé method 392
15.2 Derivation of PII 396
15.3 Solutions of PII 399
15.4 Compatibility conditions and Bäcklund transformations 402
15.5 Construction of Ψ 408
15.6 Monodromy and isomonodromy 412
15.7 The inverse problem and the Painlevé property 415
15.8 Asymptotics of PII(0) 419
15.9 Exercises 424
15.10 Remarks 428
## Contents

**Appendix A: Complex analysis**

A.1 Holomorphic and meromorphic functions 430
A.2 Cauchy’s theorem, the Cauchy integral theorem, and Liouville’s theorem 431
A.3 The residue theorem and counting zeros 432
A.4 Linear fractional transformations 434
A.5 Weierstrass factorization theorem 434
A.6 Cauchy and Stieltjes transformations and the Sokhotski–Plemelj formula 435

**Appendix B: Fourier analysis**

B.1 Fourier and inverse Fourier transforms 437
B.2 Proof of Theorem 4.1.5 438
B.3 Riemann–Lebesgue lemma 439
B.4 Fourier series and the Weierstrass approximation theorem 440
B.5 The Mellin transform and its inverse 441

*References* 443
*Author index* 463
*Notation index* 468
*Subject index* 469