

Marco Tomamichel

# Quantum Information Processing with Finite Resources

Mathematical Foundations



Springer

# Contents

<b>1</b>	<b>Introduction</b> . . . . .	1
1.1	Finite Resource Information Theory . . . . .	2
1.2	Motivating Example: Source Compression . . . . .	4
1.3	Outline of the Book . . . . .	7
<b>2</b>	<b>Modeling Quantum Information</b> . . . . .	11
2.1	General Remarks on Notation . . . . .	11
2.2	Linear Operators and Events . . . . .	13
2.2.1	Hilbert Spaces and Linear Operators . . . . .	13
2.2.2	Events and Measures . . . . .	17
2.3	Functionals and States . . . . .	18
2.3.1	Trace and Trace-Class Operators . . . . .	19
2.3.2	States and Density Operators . . . . .	20
2.4	Multi-partite Systems . . . . .	22
2.4.1	Tensor Product Spaces . . . . .	22
2.4.2	Separable States and Entanglement . . . . .	24
2.4.3	Purification . . . . .	25
2.4.4	Classical-Quantum Systems . . . . .	25
2.5	Functions on Positive Operators . . . . .	26
2.6	Quantum Channels . . . . .	28
2.6.1	Completely Bounded Maps . . . . .	28
2.6.2	Quantum Channels . . . . .	29
2.6.3	Pinching and Dephasing Channels . . . . .	30
2.6.4	Channel Representations . . . . .	31
2.7	Background and Further Reading . . . . .	32
<b>3</b>	<b>Norms and Metrics</b> . . . . .	33
3.1	Norms for Operators and Quantum States . . . . .	33
3.1.1	Schatten Norms . . . . .	34
3.1.2	Dual Norm for States . . . . .	36
3.2	Trace Distance . . . . .	37

3.3	Fidelity . . . . .	39
3.3.1	Generalized Fidelity . . . . .	41
3.4	Purified Distance . . . . .	43
3.5	Background and Further Reading . . . . .	45
<b>4</b>	<b>Quantum Rényi Divergence . . . . .</b>	<b>47</b>
4.1	Classical Rényi Divergence . . . . .	47
4.1.1	An Axiomatic Approach . . . . .	48
4.1.2	Positive Definiteness and Data-Processing . . . . .	49
4.1.3	Monotonicity in $\alpha$ and Limits . . . . .	51
4.2	Classifying Quantum Rényi Divergences . . . . .	53
4.2.1	Joint Concavity and Data-Processing . . . . .	54
4.2.2	Minimal Quantum Rényi Divergence . . . . .	55
4.2.3	Maximal Quantum Rényi Divergence . . . . .	56
4.2.4	Quantum Max-Divergence . . . . .	56
4.3	Minimal Quantum Rényi Divergence . . . . .	58
4.3.1	Pinching Inequalities . . . . .	59
4.3.2	Limits and Special Cases . . . . .	62
4.3.3	Data-Processing Inequality . . . . .	63
4.4	Petz Quantum Rényi Divergence . . . . .	67
4.4.1	Data-Processing Inequality . . . . .	67
4.4.2	Nussbaum–Szkoła Distributions . . . . .	68
4.5	Background and Further Reading . . . . .	71
<b>5</b>	<b>Conditional Rényi Entropy . . . . .</b>	<b>73</b>
5.1	Conditional Entropy from Divergence . . . . .	73
5.2	Definitions and Properties . . . . .	75
5.2.1	Alternative Expression for $\bar{H}_\alpha^1$ . . . . .	77
5.2.2	Conditioning on Classical Information . . . . .	78
5.2.3	Data-Processing Inequalities and Concavity . . . . .	79
5.3	Duality Relations and Their Applications . . . . .	80
5.3.1	Duality Relation for $\bar{H}_\alpha^1$ . . . . .	81
5.3.2	Duality Relation for $\tilde{H}_\alpha^1$ . . . . .	81
5.3.3	Duality Relation for $\bar{H}_\alpha^1$ and $\tilde{H}_\alpha^1$ . . . . .	82
5.3.4	Additivity for Tensor Product States . . . . .	83
5.3.5	Lower and Upper Bounds on Quantum Rényi Entropy . . . . .	84
5.4	Chain Rules . . . . .	86
5.5	Background and Further Reading . . . . .	89
<b>6</b>	<b>Smooth Entropy Calculus . . . . .</b>	<b>91</b>
6.1	Min- and Max-Entropy . . . . .	91
6.1.1	Semi-Definite Programs . . . . .	91
6.1.2	The Min-Entropy . . . . .	92

6.1.3	The Max-Entropy . . . . .	94
6.1.4	Classical Information and Guessing Probability . . . . .	97
6.2	Smooth Entropies . . . . .	98
6.2.1	Definition of the $\varepsilon$ -Ball . . . . .	98
6.2.2	Definition of Smooth Entropies . . . . .	99
6.2.3	Remarks on Smoothing . . . . .	100
6.3	Properties of the Smooth Entropies . . . . .	102
6.3.1	Duality Relation and Beyond . . . . .	102
6.3.2	Chain Rules . . . . .	103
6.3.3	Data-Processing Inequalities . . . . .	105
6.4	Fully Quantum Asymptotic Equipartition Property . . . . .	107
6.4.1	Lower Bounds on the Smooth Min-Entropy . . . . .	107
6.4.2	The Asymptotic Equipartition Property . . . . .	110
6.5	Background and Further Reading . . . . .	113
<b>7</b>	<b>Selected Applications . . . . .</b>	<b>115</b>
7.1	Binary Quantum Hypothesis Testing . . . . .	115
7.1.1	Chernoff Bound . . . . .	116
7.1.2	Stein's Lemma . . . . .	117
7.1.3	Hoeffding Bound and Strong Converse Exponent . . . . .	118
7.2	Entropic Uncertainty Relations . . . . .	118
7.2.1	Tripartite Uncertainty Relation . . . . .	119
7.2.2	Bipartite Uncertainty Relation . . . . .	120
7.3	Randomness Extraction . . . . .	121
7.3.1	Uniform and Independent Randomness . . . . .	121
7.3.2	Direct Bound: Leftover Hash Lemma . . . . .	123
7.3.3	Converse Bound . . . . .	125
7.4	Background and Further Reading . . . . .	126
	<b>Appendix: Some Fundamental Results in Matrix Analysis . . . . .</b>	<b>127</b>
	<b>References . . . . .</b>	<b>131</b>