Microstructured Materials: Inverse Problems
# Contents

1 **Introduction** ................................................................. 1

2 **Inverse Problems and Non-destructive Evaluation** .......................... 5
   2.1 Inverse Problems from a Mathematical Viewpoint ....................... 5
   2.2 Inverse Problems and Non-destructive Evaluation from a Practical Viewpoint ................................................................. 6
       2.2.1 General Remarks .......................................................... 6
       2.2.2 Practical Realisation .................................................... 7

3 **Mathematical Models of Microstructured Solids** ............................ 11
   3.1 Basic Principles ........................................................................ 11
   3.2 Microstructured Solids ............................................................ 12
   3.3 General Formulation of Inverse Problems .................................... 17

4 **Linear Waves** ......................................................................... 21
   4.1 Dispersion Relations. Harmonic Waves ...................................... 21
       4.1.1 Hierarchical Equation ..................................................... 21
       4.1.2 Coupled System ............................................................ 23
       4.1.3 Comparison of Models .................................................... 25
   4.2 Other Linear Waves ................................................................... 27
       4.2.1 General Solution Formula .............................................. 27
       4.2.2 Right-Propagating Waves .............................................. 28
       4.2.3 Gaussian Wave Packets .................................................. 30
   4.3 Proofs of Mathematical Statements ............................................ 32

5 **Inverse Problems for Linear Waves** ........................................... 37
   5.1 Inverse Problems for Harmonic Waves ...................................... 37
       5.1.1 Hierarchical Equation ..................................................... 37
       5.1.2 Coupled System ............................................................ 39
       5.1.3 General Consequences .................................................... 43
   5.2 Inverse Problems for Gaussian Wave Packets ................................ 43
5.3 Reconstruction of Parameters from Spectra of Waves  
5.3.1 The Case of Deformation Boundary Condition 46  
5.3.2 The Case of Displacement Boundary Condition 49  
5.4 Stability and Examples 50  
5.4.1 Stability of Solutions 50  
5.4.2 Numerical Examples 50  
5.5 Proofs of Mathematical Statements 53  
5.5.1 Proof of Theorem 5.2 53  
5.5.2 Proofs of Sect. 5.2 55  

6 Solitary Waves in Nonlinear Models 61  
6.1 Solitary Waves 61  
6.2 Solitary Wave Solutions of Hierarchical Equation 62  
6.2.1 Reduction to Equation of First Kind. Canonical Description 63  
6.2.2 Existence and Basic Properties of Canonical Waves 65  
6.2.3 Physical and Geometrical Properties of Solitary Waves in General Form 71  
6.2.4 Series Expansion of Solitary Wave 73  
6.3 Solitary Wave Solutions of Coupled System 77  
6.3.1 Separation of Unknowns. Reduction of System 77  
6.3.2 Existence and Basic Properties of Canonical Waves 81  
6.3.3 Properties of General Solitary Waves 88  
6.3.4 The Case \( \nu = 0 \) 91  
6.3.5 Comparison with Hierarchical Equation 93  
6.4 Proofs of Mathematical Statements 94  
6.4.1 Proofs of Sect. 6.2 94  
6.4.2 Proofs of Sect. 6.3 96  

7 Inverse Problems for Solitary Waves 103  
7.1 Inverse Problems for Hierarchical Equation 103  
7.1.1 Formulation of Inverse Problems 103  
7.1.2 Uniqueness Issues 105  
7.1.3 Stability Estimates 108  
7.2 Inverse Problems for Coupled System 115  
7.2.1 Formulation of Inverse Problems 115  
7.2.2 Uniqueness Issues 117  
7.3 Methods of Solution of Inverse Problems 121  
7.3.1 Minimisation of Cost Functional 121  
7.3.2 Application of Series Expansion. Linearisation 122  
7.3.3 Numerical Examples 124  
7.4 Proofs of Mathematical Statements 127  
7.4.1 Proofs of Sect. 7.1.2 127  
7.4.2 Proof of Theorem 7.5 130  
7.4.3 Proofs of Sect. 7.2.2 137
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 Summary</td>
<td>147</td>
</tr>
<tr>
<td>8.1 General Glance at Mathematical Methods</td>
<td>147</td>
</tr>
<tr>
<td>8.2 From Mathematics to Physics</td>
<td>149</td>
</tr>
<tr>
<td>8.3 Epilogue</td>
<td>153</td>
</tr>
<tr>
<td>References</td>
<td>155</td>
</tr>
<tr>
<td>Index</td>
<td>159</td>
</tr>
</tbody>
</table>