to $T_a$ is found by solving the equations

$$b_1 + b_2 a_1 = 0, \quad b_2 a_2 = 1,$$

whence

$$b_2 = \frac{1}{a_2}, \quad b_1 = -\frac{a_1}{a_2}.$$  

One can verify by substituting these values of $b_1, b_2$ in Eqs. (1.2.15) indeed $\bar{x} = x$. Thus, $T_a^{-1} = T_{a^{-1}}$, where $a^{-1}$ is the vector valued parameter

$$a^{-1} = \left( -\frac{a_1}{a_2}, \frac{1}{a_2} \right).$$

**Definition 1.4.** Let $G$ be a transformation group. Its subset $H \subseteq G$ is a subgroup of $G$, if $H$ possesses all group properties (1.2.7), i.e. $I \in H$, $T_i T_j H$ whenever $T, T_i, T_j \in H$.

**Example 1.4.** The set

$$H = \{I, T_1\}$$

is a subgroup of the group $G = \{I, T_1, \ldots, T_3\}$ of the transformations (1.2.4). Indeed $H$ contains the identity transformation $I$. Furthermore,

$$T_i^{-1}, T_i^2 \in H$$

because $T_i^2 = I$, and hence $T_i^{-1} = T_i$.

**Example 1.5.** The two-parameter group (1.2.13) has two one-parameter subgroups, namely, the translation group (1.2.10) with $a = a_1$,

$$\bar{x} = x + a,$$

and the dilation group

$$T_a : \bar{x} = ax, \quad a \neq 0,$$

with the parameter $a = a_2$. The group properties for the dilation (1.2.13) is examined as above. Namely, one can readily verify that the multiplications $T_a$ and $T_b$ yields (compare with Eq. (1.2.11))

$$T_b T_a = T_{ab}.$$

Equation (1.2.20) shows that the inverse transformation to $T_a$ is the dilation parameter $a^{-1} = 1/a$:

$$T_a^{-1} = T_{a^{-1}}.$$  

The identity transformation is obtained by letting $a = 1$.

**Definition 1.5.** Two transformation groups are said to be similar if obtained from another by an appropriate change of the variables $x'$. 
1.6. The set $G$ of transformations

\[ \bar{x} = x + \alpha_1 - \ln (1 + \alpha_2 e^x) \]

with two parameters, $\alpha_1, \alpha_2$, where $|\alpha_2|$ is sufficiently small, defines a two-parameter group ([14], Section 6.3.1). Indeed, let us rewrite the above transformation form

\[ \bar{x} = -\ln [e^{-\alpha_1} (e^{-x} + \alpha_2)] \]  

(1.2.21)

by $T_\alpha$ and $T_\beta$ two transformations (1.2.21) with the vector parameters $\alpha_2$ and $\beta = (\beta_1, \beta_2)$, respectively. Thus,

\[ T_\alpha : \bar{x} = -\ln [e^{-\alpha_1} (e^{-x} + \alpha_2)], \]
\[ T_\beta : \bar{x} = -\ln [e^{-\beta_1} (e^{-x} + \beta_2)]. \]

From (1.2.21) that $G$ contains the identical transformation and that it appears by setting

\[ \alpha_1 = \alpha_2 = 0. \]  

(1.2.22)

We have

\[ T_\beta T_\alpha : \bar{x} = -\ln [e^{-\beta_1} (e^{-x} + \beta_2)] \]
\[ = -\ln [e^{-\beta_1} (e^{\ln(e^{-x} + \alpha_2)e^{\alpha_1}} + \beta_2)]. \]

Thus, can be written

\[ = -\ln [e^{-(\alpha_1 + \beta_1)} (e^{-x} + \alpha_2 + \beta_2 e^{\alpha_1})] = -\ln [e^{-\gamma_1} (e^{-x} + \gamma_2)], \]

\[ \gamma_1 = \alpha_1 + \beta_1, \quad \gamma_2 = \alpha_2 + \beta_2 e^{\alpha_1}. \]  

(1.2.23)

\[ T_\beta T_\alpha = T_\gamma \in G, \]  

(1.2.24)

$(\gamma_1, \gamma_2)$ is defined by Eqs. (1.2.23). Equations (1.2.23) and (1.2.24) show that the parameter $\beta$ of the inverse transformation $T_\beta = T_\alpha^{-1}$ to $T_\alpha$ is obtained from (1.2.23) by letting $\gamma_1 = \gamma_2 = 0$. Hence,

\[ T_\alpha^{-1} = T_{-\gamma} \in G, \quad \text{where} \quad \alpha^{-1} = (-\alpha_1, -\alpha_2 e^{-\alpha_1}). \]  

(1.2.25)

$\gamma$ is a group. It is similar to the general linear group (1.2.13). Namely, after substitution $y = e^{-x}$ the transformation (1.2.21) is written in the form (1.2.13):

\[ \bar{y} = \alpha_1 + \alpha_2 y. \]