

Willi Freeden • Martin Gutting

Special Functions of Mathematical (Geo-)Physics

 Birkhäuser

Contents

1	Introduction: Geomathematical Motivation	1
1.1	Example: Gravitation (Laplace and Poisson Equation)	1
1.2	Example: Geomagnetism (Maxwell's Equations)	9
1.3	Example: Fluid Flow (Navier–Stokes Equation)	13
1.4	Example: Elastic Field (Cauchy–Navier Equation).....	16
Part I Auxiliary Functions		
2	The Gamma Function	25
2.1	Definition and Functional Equation	25
2.2	Euler's Beta Function	29
2.3	Stirling's Formula	33
2.4	Pochhammer's Factorial	38
2.5	Exercises (Incomplete Gamma and Beta Function, Applications in Statistics)	43
3	Orthogonal Polynomials	47
3.1	Properties of Orthogonal Polynomials	56
3.2	Quadrature Rules and Orthogonal Polynomials	66
3.3	The Jacobi Polynomials	70
3.4	Ultraspherical Polynomials	80
3.5	Application of the Legendre Polynomials in Electrostatics	90
3.6	Hermite Polynomials and Applications	95
3.7	Laguerre Polynomials and Applications	98
3.8	Exercises (Gaussian Integration, Legendre Series, Kernel Expansions)	101
Part II Spherically Oriented Functions		
4	Scalar Spherical Harmonics in \mathbb{R}^3	113
4.1	Basic Notation	114
4.2	Orthogonal Invariance.....	119

4.3	Homogeneous Polynomials on the Unit Sphere in \mathbb{R}^3	125
4.4	Closure and Completeness of Spherical Harmonics	144
4.5	The Funk–Hecke Formula and the Irreducibility of Scalar Harmonics	154
4.6	Green’s Function with Respect to the Beltrami Operator	159
4.7	The Hydrogen Atom	165
4.8	Exercises (Low Discrepancy Method, Locally Supported Wavelets, Up Function, Anharmonic Functions for the Ball, Fast Multipole Method, Wigner Matrices, Quaternionic Generation of Spherical Harmonics)	169
5	Vectorial Spherical Harmonics in \mathbb{R}^3	211
5.1	Basic Notation	212
5.2	Definition of Vector Spherical Harmonics	218
5.3	The Helmholtz Decomposition Theorem	220
5.4	Closure and Completeness of Vector Spherical Harmonics	223
5.5	Homogeneous Harmonic Vector Polynomials	231
5.6	Vectorial Beltrami Operator	236
5.7	Vectorial Addition Theorem	239
5.8	Vectorial Funk–Hecke Formulas	246
5.9	Vectorial Counterparts of the Legendre Polynomial	251
5.10	Application to Elastic Fields	254
5.11	Exercises (Uncertainty Principle, Classification of Zonal Functions, Coupling Integrals and Navier–Stokes Equation)	264
6	Spherical Harmonics in \mathbb{R}^q	285
6.1	Nomenclature and Basics	286
6.2	Integral Theorems for the Laplace Operator	290
6.3	Integral Theorems for the Laplace–Beltrami Operator	297
6.4	Homogeneous Harmonic Polynomials	302
6.5	Spherical Harmonics of Dimension q	312
6.6	Integral Theorems for the Helmholtz–Beltrami Operator	330
6.7	Exercises (Cartesian Generation of Spherical Harmonics, Best Approximations)	337
7	Classical Bessel Functions	347
7.1	Derivation and Definition of Bessel Functions	347
7.2	Orthogonality Relations	354
7.3	Bessel Functions with Integer Index	355
7.4	Exercises (Bessel Function Expansions, Hankel Transform and Discontinuous Integrals)	359
8	Bessel Functions in \mathbb{R}^q	363
8.1	Regular Bessel Functions	363
8.2	Modified Bessel Functions	370
8.3	Hankel Functions	372

8.4	Kelvin Functions	378
8.5	Expansion Theorems	385
8.6	Exercises (Helmholtz Equation, Entire Solutions, Bessel Function Like Asymptotics)	390

Part III Periodically Oriented Functions

9	Lattice Functions in \mathbb{R}	395
9.1	Bernoulli Polynomials	395
9.2	Periodic Polynomials	398
9.3	Lattice Functions	400
9.4	Euler Summation Formula	404
9.5	Riemann Zeta Function	411
9.6	Poisson Summation Formula for the Laplace Operator	415
9.7	Theta Function	419
9.8	Exercises (Trapezoidal Rule, Periodic Sobolev Spaces, Projection Method)	420
10	Lattice Functions in \mathbb{R}^q	427
10.1	Lattices in Euclidean Spaces	427
10.2	Periodic Polynomials	430
10.3	Lattice Function for the Laplace Operator	432
10.4	Euler Summation Formula for the Laplace Operator	436
10.5	Zeta Functions	439
10.6	Integral Asymptotics for Lattice Functions	443
10.7	Poisson Summation Formula	448
10.8	Theta Functions	455
10.9	Exercises (Algebraic, Periodic, and Spherical Splines, Lattice Point Sums, Lattice Point Distributions)	460
11	Concluding Remarks	483
	References	485
	Index	495