

Manfred Einsiedler • Thomas Ward

Ergodic Theory

with a view towards Number Theory

 Springer

Contents

1	Motivation	1
1.1	Examples of Ergodic Behavior	1
1.2	Equidistribution for Polynomials	3
1.3	Szemerédi's Theorem	4
1.4	Indefinite Quadratic Forms and Oppenheim's Conjecture	5
1.5	Littlewood's Conjecture	7
1.6	Integral Quadratic Forms	8
1.7	Dynamics on Homogeneous Spaces	9
1.8	An Overview of Ergodic Theory	10
2	Ergodicity, Recurrence and Mixing	13
2.1	Measure-Preserving Transformations	13
2.2	Recurrence	21
2.3	Ergodicity	23
2.4	Associated Unitary Operators	28
2.5	The Mean Ergodic Theorem	32
2.6	Pointwise Ergodic Theorem	37
2.6.1	The Maximal Ergodic Theorem	37
2.6.2	Maximal Ergodic Theorem via Maximal Inequality ...	38
2.6.3	Maximal Ergodic Theorem via a Covering Lemma	40
2.6.4	The Pointwise Ergodic Theorem	44
2.6.5	Two Proofs of the Pointwise Ergodic Theorem	45
2.7	Strong-Mixing and Weak-Mixing	48
2.8	Proof of Weak-Mixing Equivalences	54
2.8.1	Continuous Spectrum and Weak-Mixing	59
2.9	Induced Transformations	61
3	Continued Fractions	69
3.1	Elementary Properties	69
3.2	The Continued Fraction Map and the Gauss Measure	76

3.3	Badly Approximable Numbers	87
3.3.1	Lagrange's Theorem	88
3.4	Invertible Extension of the Continued Fraction Map	91
4	Invariant Measures for Continuous Maps	97
4.1	Existence of Invariant Measures	98
4.2	Ergodic Decomposition	103
4.3	Unique Ergodicity	105
4.4	Measure Rigidity and Equidistribution	110
4.4.1	Equidistribution on the Interval	110
4.4.2	Equidistribution and Generic Points	113
4.4.3	Equidistribution for Irrational Polynomials	114
5	Conditional Measures and Algebras	121
5.1	Conditional Expectation	121
5.2	Martingales	126
5.3	Conditional Measures	133
5.4	Algebras and Maps	145
6	Factors and Joinings	153
6.1	The Ergodic Theorem and Decomposition Revisited	153
6.2	Invariant Algebras and Factor Maps	156
6.3	The Set of Joinings	158
6.4	Kronecker Systems	159
6.5	Constructing Joinings	163
7	Furstenberg's Proof of Szemerédi's Theorem	171
7.1	Van der Waerden	172
7.2	Multiple Recurrence	175
7.2.1	Reduction to an Invertible System	177
7.2.2	Reduction to Borel Probability Spaces	177
7.2.3	Reduction to an Ergodic System	177
7.3	Furstenberg Correspondence Principle	178
7.4	An Instance of Polynomial Recurrence	180
7.4.1	The van der Corput Lemma	184
7.5	Two Special Cases of Multiple Recurrence	188
7.5.1	Kronecker Systems	188
7.5.2	Weak-Mixing Systems	190
7.6	Roth's Theorem	192
7.6.1	Proof of Theorem 7.14 for a Kronecker System	194
7.6.2	Reducing the General Case to the Kronecker Factor	195
7.7	Definitions	199
7.8	Dichotomy Between Relatively Weak-Mixing and Compact Extensions	201

7.9	SZ for Compact Extensions	207
7.9.1	SZ for Compact Extensions via van der Waerden	210
7.9.2	A Second Proof	212
7.10	Chains of SZ Factors	216
7.11	SZ for Relatively Weak-Mixing Extensions	218
7.12	Concluding the Proof	226
7.13	Further Results in Ergodic Ramsey Theory	227
7.13.1	Other Furstenberg Ergodic Averages	227
8	Actions of Locally Compact Groups	231
8.1	Ergodicity and Mixing	231
8.2	Mixing for Commuting Automorphisms	235
8.2.1	Ledrappier’s “Three Dots” Example	236
8.2.2	Mixing Properties of the $\times 2, \times 3$ System	239
8.3	Haar Measure and Regular Representation	243
8.3.1	Measure-Theoretic Transitivity and Uniqueness	245
8.4	Amenable Groups	251
8.4.1	Definition of Amenability and Existence of Invariant Measures	251
8.5	Mean Ergodic Theorem for Amenable Groups	254
8.6	Pointwise Ergodic Theorems and Polynomial Growth	257
8.6.1	Flows	257
8.6.2	Pointwise Ergodic Theorems for a Class of Groups	259
8.7	Ergodic Decomposition for Group Actions	266
8.8	Stationary Measures	272
9	Geodesic Flow on Quotients of the Hyperbolic Plane	277
9.1	The Hyperbolic Plane and the Isometric Action	277
9.2	The Geodesic Flow and the Horocycle Flow	282
9.3	Closed Linear Groups and Left Invariant Riemannian Metric	288
9.3.1	The Exponential Map and the Lie Algebra of a Closed Linear Group	289
9.3.2	The Left-Invariant Riemannian Metric	295
9.3.3	Discrete Subgroups of Closed Linear Groups	301
9.4	Dynamics on Quotients	305
9.4.1	Hyperbolic Area and Fuchsian Groups	306
9.4.2	Dynamics on $\Gamma \backslash \text{PSL}_2(\mathbb{R})$	310
9.4.3	Lattices in Closed Linear Groups	311
9.5	Hopf’s Argument for Ergodicity of the Geodesic Flow	314
9.6	Ergodicity of the Gauss Map	317
9.7	Invariant Measures and the Structure of Orbits	327
9.7.1	Symbolic Coding	327
9.7.2	Measures Coming from Orbits	328

10 Nilrotation	331
10.1 Rotations on the Quotient of the Heisenberg Group	331
10.2 The Nilrotation	333
10.3 First Proof of Theorem 10.1	334
10.4 Second Proof of Theorem 10.1	336
10.4.1 A Commutative Lemma; The Set K	336
10.4.2 Studying Divergence; The Set X_1	337
10.4.3 Combining Linear Divergence and the Maximal Ergodic Theorem	339
10.5 A Non-ergodic Nilrotation	341
10.6 The General Nilrotation	342
11 More Dynamics on Quotients of the Hyperbolic Plane	347
11.1 Dirichlet Regions	347
11.2 Examples of Lattices	357
11.2.1 Arithmetic and Congruence Lattices in $SL_2(\mathbb{R})$	358
11.2.2 A Concrete Principal Congruence Lattice of $SL_2(\mathbb{R})$..	358
11.2.3 Uniform Lattices	361
11.3 Unitary Representations, Mautner Phenomenon, and Ergodicity	364
11.3.1 Three Types of Actions	364
11.3.2 Ergodicity	366
11.3.3 Mautner Phenomenon for $SL_2(\mathbb{R})$	369
11.4 Mixing and the Howe–Moore Theorem	370
11.4.1 First Proof of Theorem 11.22	370
11.4.2 Vanishing of Matrix Coefficients for $PSL_2(\mathbb{R})$	372
11.4.3 Second Proof of Theorem 11.22; Mixing of All Orders .	372
11.5 Rigidity of Invariant Measures for the Horocycle Flow	378
11.5.1 Existence of Periodic Orbits; Geometric Characterization	379
11.5.2 Proof of Measure Rigidity for the Horocycle Flow	383
11.6 Non-escape of Mass for Horocycle Orbits	388
11.6.1 The Space of Lattices and the Proof of Theorem 11.32 for $X_2 = SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$	390
11.6.2 Extension to the General Case	395
11.7 Equidistribution of Horocycle Orbits	399
Appendix A: Measure Theory	403
A.1 Measure Spaces	403
A.2 Product Spaces	406
A.3 Measurable Functions	407
A.4 Radon–Nikodym Derivatives	409
A.5 Convergence Theorems	410
A.6 Well-Behaved Measure Spaces	411
A.7 Lebesgue Density Theorem	412
A.8 Substitution Rule	413

Appendix B: Functional Analysis	417
B.1 Sequence Spaces	417
B.2 Linear Functionals	418
B.3 Linear Operators	419
B.4 Continuous Functions	421
B.5 Measures on Compact Metric Spaces	422
B.6 Measures on Other Spaces	425
B.7 Vector-valued Integration	425
Appendix C: Topological Groups	429
C.1 General Definitions	429
C.2 Haar Measure on Locally Compact Groups	431
C.3 Pontryagin Duality	433
Hints for Selected Exercises	441
References	447
Author Index	463
Index of Notation	467
General Index	471