
Static & Dynamic Analysis of Structures

A Physical Approach
With Emphasis on Earthquake Engineering

Edward L. Wilson

Professor Emeritus of Civil Engineering
University of California, Berkeley

FOURTH EDITION

Revised June 2010

Computers and Structures, Inc.
Berkeley, California, USA

CONTENTS

Preface	v
Acknowledgments	vii
Personal Remarks	ix
Contents	xi
List of Tables.	xxi
List of Figures	xxv
1 Material Properties	1
1.1 Introduction	1
1.2 Anisotropic Materials.	1
1.3 Use of Material Properties within Computer Programs	3
1.4 Orthotropic Materials	4
1.5 Isotropic Materials.	5
1.6 Plane Strain Isotropic Materials	5
1.7 Plane Stress Isotropic Materials	6
1.8 Properties of Fluid-Like Materials	6
1.9 Shear and Compression Wave Velocities	7
1.10 Axisymmetric Material Properties	8
1.11 Force-Deformation Relationships	8
1.12 Summary	9
2 Equilibrium and Compatibility	11
2.1 Introduction	11
2.2 Fundamental Equilibrium Equations	12
2.3 Stress Resultants — Forces and Moments	12
2.4 Compatibility Requirements	13
2.5 Strain Displacement Equations.	13
2.6 Definition of Rotation.	14
2.7 Equations at Material Interfaces	14
2.8 Interface Equations in Finite Element Systems	15
2.9 Node Rotations in Finite Element Systems	16
2.10 Statically Determinate Structures	16
2.11 Displacement Transformation Matrix	18
2.12 Element Stiffness and Flexibility Matrices	19
2.13 Solution of Statically Determinate System	20
2.14 General Solution of Structural Systems	20

2.15	Summary	21
3	Energy and Work	23
3.1	Introduction	23
3.2	Virtual and Real Work	24
3.3	Potential Energy and Kinetic Energy	25
3.4	Strain Energy	27
3.5	External Work	28
3.6	Stationary Energy Principle	29
3.7	The Force Method	31
3.8	Lagrange's Equation of Motion	32
3.9	Conservation of Momentum	33
3.10	Summary	35
4	One-Dimensional Elements	37
4.1	Introduction	37
4.2	Analysis of an Axial Element	37
4.3	Two-Dimensional Frame Element	40
4.4	Three-Dimensional Frame Element	43
4.5	Member End-Releases	46
4.6	Summary	47
5	Isoparametric Elements	49
5.1	Introduction	49
5.2	A Simple One-Dimensional Example	50
5.3	One-Dimensional Integration Formulas	52
5.4	Restriction on Locations of Mid-Side Nodes	53
5.5	Two-Dimensional Shape Functions	54
5.6	Numerical Integration in Two Dimensions	56
5.7	Three-Dimensional Shape Functions	59
5.8	Triangular and Tetrahedral Elements	60
5.9	Summary	61
6	Incompatible Elements	63
6.1	Introduction	63
6.2	Elements with Shear Locking	64
6.3	Addition of Incompatible Modes	65
6.4	Formation of Element Stiffness Matrix	66
6.5	Incompatible Two-Dimensional Elements	67
6.6	Example using Incompatible Displacements	67
6.7	Three-Dimensional Incompatible Elements	68
6.8	Summary	69
7	Boundary Conditions and General Constraints	71
7.1	Introduction	71

7.2	Displacement Boundary Conditions	72
7.3	Numerical Problems in Structural Analysis.	73
7.4	General Theory Associated with Constraints	74
7.5	Floor Diaphragm Constraints	75
7.6	Rigid Constraints	78
7.7	Use of Constraints in Beam-Shell Analysis.	79
7.8	Use of Constraints in Shear Wall Analysis	81
7.9	Use of Constraints for Mesh Transitions.	82
7.10	Lagrange Multipliers and Penalty Functions	83
7.11	Summary	84
8	Plate Bending Elements	87
8.1	Introduction	87
8.2	The Quadrilateral Element	88
8.3	Strain-Displacement Equations	93
8.4	The Quadrilateral Element Stiffness	93
8.5	Satisfying the Patch Test.	94
8.6	Static Condensation	95
8.7	Triangular Plate Bending Element	95
8.8	Other Plate Bending Elements	95
8.9	Numerical Examples	96
8.9.1	One Element Beam.	96
8.9.2	Point Load on Simply Supported Square Plate	97
8.9.3	Uniform Load on Simply Supported Square Plate	99
8.9.4	Evaluation of Triangular Plate Bending Elements	99
8.9.5	Use of Plate Element to Model Torsion in Beams	100
8.10	Summary	101
9	Membrane Element with Normal Rotations	103
9.1	Introduction	103
9.2	Basic Assumptions	104
9.3	Displacement Approximation	105
9.4	Introduction of Node Rotation	106
9.5	Strain-Displacement Equations	107
9.6	Stress-Strain Relationship	107
9.7	Transform Relative to Absolute Rotations	107
9.8	Triangular Membrane Element.	109
9.9	Numerical Example	109
9.10	Summary	110
10	Shell Elements	111
10.1	Introduction	111
10.2	A Simple Quadrilateral Shell Element	112
10.3	Modelling Curved Shells with Flat Elements	113
10.4	Triangular Shell Elements	114

10.5	Use of Solid Elements for Shell Analysis	114
10.6	Analysis of the Scordelis-Lo Barrel Vault	114
10.7	Hemispherical Shell Example	116
10.8	Summary	117
11	Geometric Stiffness and P-Delta Effects	119
11.1	Definition of Geometric Stiffness	119
11.2	Approximate Buckling Analysis	121
11.3	P-Delta Analysis of Buildings	122
11.4	Equations for Three-Dimensional Buildings	124
11.5	The Magnitude of P-Delta Effects	125
11.6	P-Delta Analysis Without Computer Program Modification	126
11.7	Effective Length Factors — (<i>K</i> -Factors)	127
11.8	General Formulation of Geometric Stiffness	127
11.9	Summary	128
12	Dynamic Analysis	131
12.1	Introduction	131
12.2	Dynamic Equilibrium	132
12.3	Step-By-Step Solution Method	133
12.4	Mode Superposition Method	134
12.5	Response Spectra Analysis	134
12.6	Solution in the Frequency Domain	135
12.7	Solution of Linear Equations	135
12.8	Undamped Harmonic Response	136
12.9	Undamped Free Vibrations	137
12.10	Summary	137
13	Dynamic Analysis using Mode Superposition	139
13.1	Equations to be Solved	139
13.2	Transformation to Modal Equations	140
13.3	Response Due to Initial Conditions Only	141
13.4	General Solution Due to Arbitrary Loading	141
13.5	Solution for Periodic Loading	146
13.6	Participating Mass Ratios	147
13.7	Static Load Participation Ratios	148
13.8	Dynamic Load Participation Ratios	149
13.9	Summary	150
14	Calculation of Stiffness and Mass Orthogonal Vectors	153
14.1	Introduction	153
14.2	Determinate Search Method	154
14.3	Sturm Sequence Check	154
14.4	Inverse Iteration	154
14.5	Gram-Schmidt Orthogonalization	156

14.6	Block Subspace Iteration	156
14.7	Solution of Singular Systems	156
14.8	Generation of Load-Dependent Ritz Vectors	157
14.9	A Physical Explanation of the LDR Algorithm	158
14.10	Comparison of Solutions Using Eigen and Ritz Vectors	160
14.11	Correction for Higher Mode Truncation	161
14.12	Vertical Direction Seismic Response	163
14.13	Summary	165
15	Dynamic Analysis using Response Spectrum Seismic Loading	167
15.1	Introduction	167
15.2	Definition of a Response Spectrum	168
15.3	Calculation of Modal Response	169
15.4	Typical Response Spectrum Curves	169
15.5	The CQC Method of Modal Combination	172
15.6	Numerical Example of Modal Combination	173
15.7	Design Spectra	176
15.8	Orthogonal Effects in Spectral Analysis	176
15.8.1	Basic Equations for Calculation of Spectral Forces	178
15.8.2	The General CQC3 Method	179
15.8.3	Examples of Three-Dimensional Spectra Analyses	180
15.8.4	Recommendations on Orthogonal Effects	182
15.9	Limitations of the Response Spectrum Method	183
15.9.1	Story Drift Calculations	183
15.9.2	Estimation of Spectra Stresses in Beams	183
15.9.3	Design Checks for Steel and Concrete Beams	184
15.9.4	Calculation of Shear Force in Bolts	184
15.10	Summary	185
16	Soil-Structure Interaction	187
16.1	Introduction	187
16.2	Site Response Analysis	188
16.3	Kinematic or Soil Structure Interaction	188
16.4	Response Due to Multi-Support Input Motions	191
16.5	Analysis of Gravity Dam and Foundation	193
16.6	The Massless Foundation Approximation	195
16.7	Approximate Radiation Boundary Conditions	195
16.8	Use of Springs at the Base of a Structure	197
16.9	Summary	197
17	Seismic Analysis Modelling to Satisfy Building Codes	199
17.1	Introduction	199
17.2	Three-Dimensional Computer Model	200
17.3	Three-Dimensional Mode Shapes and Frequencies	201

17.4	Three-Dimensional Dynamic Analysis	204
17.4.1	Dynamic Design Base Shear.	204
17.4.2	Definition of Principal Directions	206
17.4.3	Directional and Orthogonal Effects	206
17.4.4	Basic Method of Seismic Analysis.	206
17.4.5	Scaling of Results	207
17.4.6	Dynamic Displacements and Member Forces	207
17.4.7	Torsional Effects	207
17.5	Numerical Example	208
17.6	Dynamic Analysis Method Summary	209
17.7	Summary	210
18	Fast Nonlinear Analysis	213
18.1	Introduction	213
18.2	Structures with a Limited Number of Nonlinear Elements	214
18.3	Fundamental Equilibrium Equations	214
18.4	Calculation of Nonlinear Forces	216
18.5	Transformation to Modal Coordinates	216
18.6	Solution of Nonlinear Modal Equations	218
18.7	Static Nonlinear Analysis of Frame Structure	218
18.8	Dynamic Nonlinear Analysis of Frame Structure	221
18.9	Seismic Analysis of Elevated Water Tank	224
18.10	Summary	225
19	Linear Viscous Damping	227
19.1	Introduction	227
19.2	Energy Dissipation in Real Structures	228
19.3	Physical Interpretation of Viscous Damping	229
19.4	Modal Damping Violates Dynamic Equilibrium	229
19.5	Numerical Example	230
19.6	Stiffness and Mass Proportional Damping	231
19.7	Calculation of Orthogonal Damping Matrices	232
19.8	Structures with Non-Classical Damping.	233
19.9	Nonlinear Energy Dissipation	233
19.10	Summary	234
20	Dynamic Analysis using Numerical Integration	235
20.1	Introduction	235
20.2	Newmark Family of Methods	236
20.3	Stability of Newmark's Method	237
20.4	The Average Acceleration Method	239
20.5	Wilson's θ -Factor	239
20.6	The Use of Stiffness Proportional Damping	240
20.7	The Hilber, Hughes and Taylor α -Method	241
20.8	Selection of a Direct Integration Method	241

20.9	Nonlinear Analysis	242
20.10	Summary	242
21	Nonlinear Elements	243
21.1	Introduction	243
21.2	General Three-Dimensional Two-Node Element	244
21.3	General Plasticity Element	245
21.4	Different Positive and Negative Properties	246
21.5	The Bilinear Tension-Gap-Yield Element	247
21.6	Nonlinear Gap-Crush Element	247
21.7	Viscous Damping Elements	249
21.8	Three-Dimensional Friction-Gap Element	251
21.9	Summary	252
22	Seismic Analysis using Displacement Loading	253
22.1	Introduction	253
22.2	Equilibrium Equations for Displacement Input	254
22.3	Use of Pseudo-Static Displacements	256
22.4	Solution of Dynamic Equilibrium Equations	257
22.5	Numerical Example	258
22.5.1	Example Structure	258
22.5.2	Earthquake Loading	260
22.5.3	Effect of Time Step Size for Zero Damping	260
22.5.4	Earthquake Analysis with Finite Damping	262
22.5.5	The Effect of Mode Truncation	265
22.6	Use of Load Dependent Ritz Vectors	266
22.7	Solution Using Step-By-Step Integration	267
22.8	Summary	269
23	Fluid-Structure Interaction	271
23.1	Introduction	271
23.2	Fluid-Structure Interaction	272
23.3	Finite Element Model of Dam-Foundation Interface	274
23.4	Loading Due to Uplift and Pore Water Pressure	275
23.5	Pore Water Pressure Calculation Using SAP2000	277
23.6	Selection of Gap Element Stiffness Value	279
23.7	Fundamental Equations in Fluid Dynamics	279
23.8	Relationship Between Pressure and Velocity	280
23.9	Equilibrium at the Interface of Two Materials	281
23.10	Radiation Boundary Conditions	283
23.11	Surface Sloshing Modes	283
23.12	Vertical Wave Propagation	284
23.13	The Westergaard Paper	285
23.14	Dynamic Analysis of Rectangular Reservoir	286
23.15	Energy Absorptive Reservoir Boundaries	289

23.16	Relative Versus Absolute Formulation	290
23.17	The Effect of Gate Setback on Pressure	291
23.18	Seismic Analysis of Radial Gates	293
23.19	Final Remarks	296

APPENDICES **299**

A Vector Notation **301**

A.1	Introduction	301
A.2	Vector Cross Product	302
A.3	Vectors to Define a Local Reference System	304
A.4	Fortran Subroutines for Vector Operations	305

B Matrix Notation **309**

B.1	Introduction	309
B.2	Definition of Matrix Notation	309
B.3	Matrix Transpose and Scalar Multiplication	312
B.4	Definition of a Numerical Operation	313
B.5	Programming Matrix Multiplication	313
B.6	Order of Matrix Multiplication	314
B.7	Summary	314

C Solution or Inversion of Linear Equations **317**

C.1	Introduction	317
C.2	Numerical Example	318
C.3	The Gauss Elimination Algorithm	319
C.4	Solution of a General Set of Linear Equations	321
C.5	Alternative to Pivoting	321
C.6	Matrix Inversion	323
C.7	Physical Interpretation of Matrix Inversion	325
C.8	Partial Gauss Elimination, Static Condensation and Substructure Analysis	326
C.9	Equations Stored in Banded or Profile Form	328
C.10	LDL Factorization	329
	C.10.1 Triangularization or Factorization of the A Matrix	329
	C.10.2 Forward Reduction of the b Matrix	330
	C.10.3 Calculation of x by Back substitution	331
C.11	Diagonal Cancellation and Numerical Accuracy	332
C.12	Summary	332

D The Eigenvalue Problem **335**

D.1	Introduction	335
D.2	The Jacobi Method	336
D.3	Calculation of 3D Principal Stresses	338
D.4	Solution of the General Eigenvalue Problem	339

D.5	Summary	340
E	Transformation of Material Properties	341
E.1	Introduction	341
E.2	Summary	343
F	A Displacement-Based Beam Element with Shear Deformations	345
F.1	Introduction	345
F.2	Basic Assumptions	345
F.3	Effective Shear Area	348
G	Numerical Integration	351
G.1	Introduction	351
G.2	One-Dimensional Gauss Quadrature	352
G.3	Numerical Integration in Two Dimensions	354
G.4	An Eight-Point Two-Dimensional Rule	354
G.5	An Eight-Point Lower-Order Rule	356
G.6	A Five-Point Integration Rule	356
G.7	Three-Dimensional Integration Rules	357
G.8	Selective Integration	359
G.9	Summary	360
H	Speed of Computer Systems	363
H.1	Introduction	363
H.2	Definition of One Numerical Operation	363
H.3	Speed of Different Computer Systems	364
H.4	Speed of Personal Computer Systems	364
H.5	Paging Operating Systems	365
H.6	Summary	366
I	Method of Least Square	367
I.1	Simple Example	367
I.2	General Formulation	368
I.3	Calculation of Stresses within Finite Elements	370
J	Consistent Earthquake Acceleration and Displacement Records	373
J.1	Introduction	373
J.2	Ground Acceleration Records	374
J.3	Calculation of Acceleration Record from Displacement Record	375
J.4	Creating Consistent Acceleration Record	377
J.5	Summary	378
K	Example of a SAP2000 Model for a Slab/Wall System	381
K.1	Introduction	381
K.2	The Problem.	382

K.3	A Shell/Beam Model of the Slab/Wall System.	382
K.4	Shell Element Modification	384
Bibliography	385
Index	389