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Discrete Weak KAM Theory

An Introduction through Examples and its
Applications to Twist Maps

 Springer

Contents

1	Introduction	1
1.1	Weak KAM, a Bridge Between Aubry–Mather and Viscosity Solutions	2
1.1.1	Aubry–Mather Theory	2
1.1.2	Viscosity Solutions	3
1.1.3	The Bridge	4
1.2	Weak KAM, Beyond Hamilton–Jacobi Equations	6
1.2.1	Optimal Control Theory	6
1.2.2	Contact Type and Systems of Hamilton–Jacobi Equations ...	7
1.2.3	Lorentzian Geometry and Lyapunov Functions	7
1.2.4	Optimal Transportation	8
1.2.5	Discrete Weak KAM Theory, with an Economical Twist	9
1.3	Organisation of the Text	10
2	The Discrete Setting, Weak KAM Solutions and Subsolutions	13
2.1	Discrete Setting and the Lax–Oleinik Semigroup	13
2.2	The Weak KAM Theorem and Critical Subsolutions	15
2.3	The Positive Lax–Oleinik Semigroup	19
2.4	Strict Subsolutions, Aubry Sets	20
2.5	Relations to the Classical Theory	26
2.5.1	Classical Setting and Lax–Oleinik Semigroup	26
2.5.2	The Weak KAM Theorem and Critical Subsolutions	30
2.5.3	The Positive Lax–Oleinik Semigroup	34
2.5.4	Strict Subsolutions, Aubry Sets	35
3	More (Dynamical) Characterizations of the Aubry Sets	39
3.1	The Peierls Barrier	39
3.2	Examples of Points in the Aubry Sets	44
3.3	Regularity of Subsolutions	46
3.4	More Regularity for Subsolutions	47

3.5	Graph Properties and Dynamics on the Aubry Set	51
3.6	Relations to the Classical Theory	52
3.6.1	The Classical Peierls Barrier	52
3.6.2	Examples of Points in the Aubry Set	54
3.6.3	Regularity and More Regularity of Subsolutions	55
3.6.4	Graph Properties, Twist Condition and Dynamics on the Aubry Set	56
4	Minimizing Mather Measures and the Discounted Semigroups	59
4.1	Minimizing Mather Measures	59
4.1.1	An Optimal Transport Like Approach	60
4.1.2	An Ergodic Point of View	63
4.2	The Discounted Equation	64
4.3	Discount for the Positive Lax–Oleinik Semigroup	70
4.4	Degenerate Discounted Equations	73
4.5	Comment on the Discounted Procedure	83
4.6	Relations to the Classical Theory	86
4.6.1	Minimizing Mather Measures	87
4.6.2	The Classical Discounted Equation	92
4.6.3	Discount for the Positive Classical L.–O. Semigroup	93
4.6.4	Some Degenerate Discounted Hamilton–Jacobi Equations...	94
5	A Family of Examples	97
5.1	The Study of H_0	98
5.2	Increasing the Cohomology Class: $c \in [0, \alpha]$	99
5.3	A Change of Regime: $c \in (\alpha, \int_0^1 f^+(x)dx)$	102
5.4	The Limiting Case: $c_0 = \int_0^1 f^+(x)dx$	103
5.5	Positive Rotation Numbers: $c > \int_0^1 f^+(x)dx$	104
5.5.1	Non-continuity of u_1^c with Respect to c	106
5.5.2	A Situation Where $u_1^c \neq U_0^c$	111
5.6	Concluding Example	114
6	Twist Maps	115
6.1	Definitions and Variational Structure	115
6.1.1	Definition and Birkhoff’s Theorem	116
6.1.2	The Generating Function, Properties and Consequences	117
6.2	Examples and Moser’s Theorem	124
6.2.1	Notions of Integrability	124
6.2.2	The Standard Family	126
6.2.3	General Twist Maps and a Theorem of Moser	126
6.3	Weak KAM for Twist Maps	128

6.4	Mather Measures	136
6.5	Order Properties of Weak KAM Solutions	141
6.6	Structure of Infinite Minimizing Chains	147
6.6.1	Irrational Rotation Number.....	151
6.6.2	Rational Rotation Number	153
6.7	A Glimpse into the World of Weakly Integrable Twist Maps	161
Epilogue: Weak KAM Theory, Origins and Present Developments		165
	Etymology of Weak KAM	165
	Arnol'd Diffusion	167
	The Mañé Conjecture and the Size of the Aubry Set	168
	The Shape of the Aubry Set.....	169
	Singularities of Weak KAM Solutions	170
	Contact Type Hamilton–Jacobi Equations	171
	Weak KAM in Celestial Mechanics	173
References		175
Index		185