

# Mathematical Modeling

## Models, Analysis and Applications Second Edition

Sandip Banerjee

Indian Institute of Technology Roorkee, India



CRC Press

Taylor & Francis Group

Boca Raton London New York

---

CRC Press is an imprint of the  
Taylor & Francis Group, an **informa** business  
A CHAPMAN & HALL BOOK

---

# **Contents**

<b>Foreword</b>	<b>xiii</b>
<b>Preface to Second Edition</b>	<b>xv</b>
<b>Author</b>	<b>xvii</b>
<b>1 About Mathematical Modeling</b>	<b>1</b>
1.1 What Is Mathematical Modeling? . . . . .	1
1.2 History of Mathematical Modeling . . . . .	1
1.3 Importance of Mathematical Modeling . . . . .	3
1.4 Latest Developments in Mathematical Modeling . . . . .	4
1.5 Limitations of Mathematical Modeling . . . . .	6
1.6 Units . . . . .	7
1.7 Dimensions . . . . .	8
1.8 Dimensional Analysis . . . . .	11
1.9 Scaling . . . . .	13
1.10 How to Built Mathematical Models . . . . .	16
1.10.1 Step I (The Start) . . . . .	16
1.10.2 Step II (The Assumption) . . . . .	16
1.10.3 Step III (Schematic or Flow Diagrams) . . . . .	18
1.10.4 Step IV (Choosing Mathematical Equations) . . . . .	18
1.10.5 Step V (Solving Equations) . . . . .	18
1.10.6 Step VI (Interpretation of the Result) . . . . .	19
1.11 Mathematical Models and Functions . . . . .	19
1.11.1 Linear Models . . . . .	19
1.11.2 Quadratic Models . . . . .	21
1.11.3 Cubic Models . . . . .	22
1.11.4 Logistic Function and Logistic Growth Model . . . . .	23
1.11.5 Gompertz Function and Gompertz Growth Model . . . . .	23
1.12 Functional Responses in Population Dynamics . . . . .	24
1.12.1 Holling Type I Functional Response . . . . .	24
1.12.2 Holling Type II Functional Response . . . . .	25
1.12.3 Holling Type III Functional Response . . . . .	26
1.13 Miscellaneous Examples . . . . .	27
1.14 Exercises . . . . .	30

<b>2 Discrete Models Using Difference Equations</b>	<b>35</b>
2.1 Difference Equations . . . . .	35
2.1.1 Linear Difference Equation with Constant Coefficients	36
2.1.2 Solution of Homogeneous Equations . . . . .	36
2.1.3 Difference Equations: Equilibria and Stability . . . . .	41
2.1.3.1 Linear Difference Equations . . . . .	41
2.1.3.2 System of Linear Difference Equations . . . . .	43
2.1.3.3 Non-Linear Difference Equations . . . . .	47
2.2 Introduction to Discrete Models . . . . .	49
2.3 Linear Models . . . . .	50
2.3.1 Population Model Involving Growth . . . . .	50
2.3.2 Newton's Law of Cooling . . . . .	52
2.3.3 Bank Account Problem . . . . .	53
2.3.4 Drug Delivery Problem . . . . .	56
2.3.5 Harrod Model (Economic Model) . . . . .	58
2.3.6 Arms Race Model . . . . .	58
2.3.7 Lanchester's Combat Model . . . . .	59
2.3.8 Linear Predator–Prey Model . . . . .	60
2.4 Non-Linear Models . . . . .	63
2.4.1 Density-Dependent Growth Models . . . . .	63
2.4.1.1 Logistic Model . . . . .	63
2.4.1.2 Richer's Model . . . . .	64
2.4.2 The Learning Model . . . . .	65
2.4.3 Dynamics of Alcohol: A Mathematical Model . . . . .	65
2.4.4 Two Species Competition Model . . . . .	68
2.4.5 2-cycles . . . . .	71
2.4.6 Stability of 2-cycles . . . . .	71
2.4.7 3-cycles . . . . .	73
2.5 Bifurcations in Discrete Models . . . . .	75
2.6 Chaos in Discrete Models . . . . .	80
2.6.1 Criteria of Chaos for Discrete Dynamical System . . . . .	80
2.6.2 Quantification of Chaos: Lyapunov Exponent . . . . .	81
2.7 Miscellaneous Examples . . . . .	84
2.8 Mathematica Codes . . . . .	105
2.8.1 Lanchester's Combat Model (Figure 2.12(a)) . . . . .	105
2.8.2 Lyapunov Exponent (Figure 2.20(a)) . . . . .	105
2.8.3 Two Species Competition Model (Figure 2.16(a)) . . . . .	106
2.8.4 Saddle-Node Bifurcation (Figure 2.18(a)) . . . . .	106
2.8.5 Neimark-Sacker Bifurcation (Figure 2.19) . . . . .	107
2.9 Matlab Codes . . . . .	107
2.9.1 Lanchester's Combat Model (Figure 2.12(a)) . . . . .	107
2.9.2 Saddle-Node Bifurcation (Figure 2.18(a)) . . . . .	108
2.9.3 Chaotic Behavior in Tent Map (Figure 2.20(b)) . . . . .	108
2.9.4 2D Bifurcation (Figure 2.31) . . . . .	109
2.10 Exercises . . . . .	109
2.11 Projects . . . . .	120

<b>3 Continuous Models Using Ordinary Differential Equations</b>	<b>123</b>
3.1 Introduction to Continuous Models . . . . .	123
3.2 Steady-State Solution . . . . .	125
3.3 Stability . . . . .	126
3.3.1 Linearization and Local Stability Analysis . . . . .	127
3.3.2 Lyapunov's Direct Method . . . . .	130
3.3.2.1 Lyapunov's Condition for Local Stability . .	130
3.3.2.2 Lyapunov's Condition for Global Stability . .	131
3.4 Phase Plane Diagrams of Linear Systems . . . . .	132
3.5 Continuous Models . . . . .	138
3.5.1 Carbon Dating . . . . .	138
3.5.2 Drug Distribution in the Body . . . . .	140
3.5.3 Growth and Decay of Current in an L-R Circuit .	141
3.5.4 Rectilinear Motion under Variable Force . . . . .	143
3.5.5 Mechanical Oscillations . . . . .	144
3.5.5.1 Horizontal Oscillations . . . . .	144
3.5.5.2 Vertical Oscillations . . . . .	145
3.5.5.3 Damped and Forced Oscillations . . . . .	146
3.5.6 Dynamics of Rowing . . . . .	148
3.5.7 Arms Race Models . . . . .	149
3.5.8 Epidemic Models . . . . .	152
3.5.9 Combat Models . . . . .	157
3.5.9.1 Conventional Combat Model . . . . .	158
3.5.9.2 Guerrilla Combat Model . . . . .	159
3.5.9.3 Mixed Combat Model . . . . .	159
3.5.9.4 Guerrilla Combat Model (Revisited) . . . . .	160
3.5.10 Mathematical Model of Love Affair . . . . .	163
3.6 Bifurcations . . . . .	167
3.6.1 Bifurcations in One-Dimension . . . . .	167
3.6.1.1 Saddle-Node Bifurcation . . . . .	167
3.6.1.2 Transcritical Bifurcation . . . . .	168
3.6.1.3 Pitchfork Bifurcation . . . . .	169
3.6.2 Bifurcation in Two-Dimensions . . . . .	170
3.6.2.1 Sotomayor's Theorem . . . . .	171
3.6.2.2 Saddle-Node Bifurcation . . . . .	172
3.6.2.3 Transcritical Bifurcation . . . . .	172
3.6.2.4 Pitchfork Bifurcation . . . . .	173
3.6.2.5 Hopf Bifurcation . . . . .	175
3.7 Estimation of Model Parameters . . . . .	177
3.7.1 Least Squares Method . . . . .	177
3.7.2 Fitting a Suitable Curve to the Given Data . . . . .	179
3.7.3 Parameter Estimation for ODE . . . . .	183
3.8 Chaos in Continuous Models . . . . .	185
3.8.1 Lyapunov Exponents . . . . .	186
3.8.2 Rössler Systems: Equations for Continuous Chaos . .	187

3.9	Miscellaneous Examples . . . . .	190
3.10	Mathematica Codes . . . . .	211
3.10.1	Stable Node (Figure 3.1(a)) . . . . .	211
3.10.2	Stable Node (Figure 3.1(b)) . . . . .	211
3.10.3	One-Dimensional Bifurcation (Figure 3.43(a)) . . . . .	212
3.10.4	Chaotic Behavior (Figure 3.34(c)) . . . . .	212
3.11	Matlab Codes . . . . .	212
3.11.1	Stable Node (Figure 3.1(a)) . . . . .	213
3.11.2	Stable Node (Figure 3.1(b)) . . . . .	213
3.11.3	Saddle Node Bifurcation (Figure 3.22(a)) . . . . .	214
3.11.4	Chaotic Behavior (Figure 3.34(c)) . . . . .	214
3.12	Exercises . . . . .	215
3.13	Projects . . . . .	241
4	<b>Spatial Models Using Partial Differential Equations</b>	<b>243</b>
4.1	Introduction . . . . .	243
4.2	Heat Flow through a Small Thin Rod (One Dimensional) . . . . .	246
4.3	Two-Dimensional Heat-Equation (Diffusion Equation) . . . . .	252
4.4	Steady Heat Flow: Laplace Equation . . . . .	255
4.4.1	Laplace Equation with Dirichlet's Boundary Condition	255
4.4.2	Laplace Equation with Neumann's Boundary Condition	258
4.5	Wave Equation . . . . .	258
4.5.1	Vibrating String . . . . .	259
4.6	Two-Dimensional Wave Equation . . . . .	262
4.7	Fluid Flow through a Porous Medium . . . . .	266
4.8	Traffic Flow . . . . .	266
4.9	Crime Model . . . . .	270
4.10	Reaction-Diffusion Systems . . . . .	272
4.10.1	Population Dynamics with Diffusion (Single Species) . . . . .	273
4.10.2	Population Dynamics with Diffusion (Two Species) . . . . .	275
4.11	Mathematica Codes . . . . .	282
4.11.1	Heat Equation with Dirichlet's Condition (Figure 4.2)	282
4.11.2	Heat Equation with Neumann's condition (Figure 4.3)	282
4.11.3	One-Dimensional Wave Equation (Figure 4.7) . . . . .	283
4.11.4	Two-Dimensional Heat Equation (Figure 4.4) . . . . .	283
4.11.5	Brusselator Equation One-Dimensional (Figure 4.12) . . . . .	284
4.11.6	Brusselator Equation Two-Dimensional (Figure 4.13) . . . . .	284
4.12	Matlab Codes . . . . .	285
4.12.1	Heat Flow (Figure 4.2) . . . . .	285
4.12.2	Brusselator Equation (Figure 4.12) . . . . .	285
4.12.3	Wave Equation (Figure 4.7(a)) . . . . .	286
4.13	Miscellaneous Examples . . . . .	287
4.14	Exercises . . . . .	295
4.15	Project . . . . .	305

<b>5 Modeling with Delay Differential Equations</b>	<b>307</b>
5.1 Introduction . . . . .	307
5.2 Linear Stability Analysis . . . . .	308
5.2.1 Linear Stability Criteria . . . . .	309
5.3 Different Models with Delay Differential Equations . . . . .	311
5.3.1 Delayed Protein Degradation . . . . .	311
5.3.2 Football Team Performance Model . . . . .	312
5.3.3 Shower Problem . . . . .	313
5.3.4 Breathing Model . . . . .	314
5.3.5 Housefly Model . . . . .	315
5.3.6 Two-Neuron System . . . . .	316
5.4 Immunotherapy with Interleukin-2, a Study Based on Mathematical Modeling [11] (a Research Problem) . . . . .	317
5.4.1 Background of the Problem . . . . .	317
5.4.2 The Model . . . . .	318
5.4.3 Positivity of the Solution . . . . .	321
5.4.4 Linear Stability Analysis with Delay . . . . .	321
5.4.5 Delay Length Estimation to Preserve Stability . . . . .	324
5.4.6 Numerical Results . . . . .	326
5.4.7 Conclusion . . . . .	328
5.5 Miscellaneous Examples . . . . .	329
5.6 Mathematica Codes . . . . .	337
5.6.1 Delayed Protein Degradation (Figure 5.1) . . . . .	337
5.6.2 Two Neuron System (Figure 5.6) . . . . .	337
5.7 Matlab Codes . . . . .	338
5.7.1 Delayed Protein Degradation (Figure 5.1) . . . . .	338
5.7.2 Two Neuron System (Figure 5.6) . . . . .	338
5.8 Exercises . . . . .	339
5.9 Project . . . . .	345
<b>6 Modeling with Stochastic Differential Equations</b>	<b>347</b>
6.1 Introduction . . . . .	347
6.1.1 Random Experiment . . . . .	347
6.1.2 Outcome . . . . .	347
6.1.3 Event . . . . .	347
6.1.4 Sample Space . . . . .	348
6.1.5 Event Space . . . . .	348
6.1.6 Axiomatic definition of Probability . . . . .	348
6.1.7 Probability Function . . . . .	348
6.1.8 Probability Space . . . . .	348
6.1.9 Random Variable . . . . .	349
6.1.10 Sigma Algebra . . . . .	349
6.1.11 Measure . . . . .	349
6.1.12 Probability Measure . . . . .	350
6.1.13 Mean and Variance . . . . .	350

6.1.14	Independent Random Variables . . . . .	350
6.1.15	Gaussian Distribution (Normal Distribution) . . . . .	351
6.1.16	Characteristic Function . . . . .	351
6.1.17	Characteristic Function of Gaussian Distribution . . . . .	351
6.1.18	Inversion Theorem . . . . .	352
6.1.19	Convergence of Random Variables and Limit Theorems	352
6.1.20	Stochastic Process . . . . .	354
6.1.21	Markov Process . . . . .	355
6.1.22	Gaussian (Normal) Process . . . . .	356
6.1.23	Wiener Process (Brownian Motion) . . . . .	356
6.1.24	Gaussian White Noise . . . . .	356
6.1.25	Ito Integral: . . . . .	357
6.1.26	Ito's Formula in One-Dimension . . . . .	359
6.1.27	Stochastic Differential Equation (SDE) . . . . .	359
6.1.28	Stochastic Stability . . . . .	360
6.1.28.1	Stable in Probability . . . . .	360
6.1.28.2	Almost sure Exponential Stability . . . . .	361
6.1.28.3	Moment Exponential Stability . . . . .	361
6.2	Stochastic Models . . . . .	361
6.2.1	Stochastic Logistic Growth . . . . .	361
6.2.2	RLC Electric Circuit with Randomness . . . . .	362
6.2.3	Heston Model . . . . .	363
6.2.4	Two Species Stochastic Competition Model . . . . .	363
6.3	Research Problem: Cancer Self-Remission and Tumor Stability—a Stochastic Approach [145] . . . . .	366
6.3.1	Background of the Problem . . . . .	366
6.3.2	The Deterministic Model . . . . .	367
6.3.3	Equilibria and Local Stability Analysis . . . . .	368
6.3.4	Biological Implications . . . . .	370
6.3.5	The Stochastic Model . . . . .	371
6.3.6	Stochastic Stability of the Positive Equilibrium . . . . .	373
6.3.7	Numerical Results and Biological Interpretations . . . . .	376
6.4	Mathematica Codes . . . . .	378
6.4.1	Stochastic Logistic Growth (Figure 6.2) . . . . .	378
6.4.2	Stochastic Competition Model (Figure 6.4) . . . . .	379
6.5	Matlab Codes . . . . .	379
6.5.1	Stochastic Logistic Growth (Figure 6.2) . . . . .	379
6.5.2	Stochastic Competition Model (Figure 6.4(c)) . . . . .	380
6.6	Exercises . . . . .	381
<b>7</b>	<b>Hints and Solutions</b>	<b>385</b>
<b>Bibliography</b>		<b>401</b>
<b>Index</b>		<b>413</b>