

# **How to Derive a Formula**

**Volume 1:  
Basic Analytical Skills and  
Methods for Physical Scientists**

# Contents

<i>Preface</i>	ix
<i>About the Authors</i>	xvii
<i>Acknowledgements</i>	xxi
<i>Introduction</i>	xxiii

## **Part I: From Base Camp—Understanding Functions and Variables: The First Stage** 1

1. Essential Functions	3
1.1 Polynomial and Rational Functions . . . . .	3
1.2 Transcendental Functions . . . . .	16
1.3 Inverse Functions and Branches . . . . .	22
1.4 Key Points . . . . .	29
2. Polynomial Expansions: When They Work and When They Don't	31
2.1 Maclaurin Expansions . . . . .	31
2.2 Taylor Expansions . . . . .	42
2.3 When Maclaurin and Taylor Expansions Don't Work . . . . .	46
2.4 Life Without a Calculator . . . . .	47

2.5	Key Points . . . . .	48
3.	Limits, Differentiation and Integration	51
3.1	Limits . . . . .	51
3.2	Differentiation . . . . .	59
3.3	Integration . . . . .	67
3.4	Key Points . . . . .	79
4.	The Way to Check Yourself: Analysis of Limiting Cases	81
4.1	Key Points . . . . .	92
5.	Definite Integrals as Functions	93
5.1	Warm Up . . . . .	93
5.2	Integrals as Functions of a Variable in the Integrand . . . . .	94
5.3	Integrals as Functions of the Limit of Integration . . . . .	101
5.4	Key Points . . . . .	103
6.	Probability Distribution Functions, and Filter Functions as Limiting Cases	105
6.1	Probability and the Binomial Distribution . . . . .	106
6.2	An Introduction to Continuous Distribution Functions . . . . .	110
6.3	Expectation Values and Standard Deviation . . . . .	116
6.4	Classic Examples of the Distribution Function . . . . .	120
6.5	Conditional Probability as the Convolution of Two Distribution Functions . . . . .	128
6.6	Defining the Dirac Delta Function and Heaviside Step Function as Limits . . . . .	131
6.7	Key Points . . . . .	133

7. Vectors and Introduction to Vector Calculus	135
7.1 Vector Basics . . . . .	135
7.2 Vector Coordinate Geometry 1 . . . . .	143
7.3 Elementary Vector Calculus . . . . .	153
7.4 Key Points . . . . .	170
8. Understanding Sequences and Series	173
8.1 Basic Notions . . . . .	173
8.2 Calculating Exactly and Estimating the Sums of Series . . . . .	177
8.3 The Integral Test and Conditionally Converging Series . . . . .	186
8.4 Examples of Sums in Physics . . . . .	191
8.5 Key Points . . . . .	199
9. Complex Numbers	201
9.1 What is a Complex Number? . . . . .	201
9.2 Exponential Functions with Imaginary and Complex Arguments . . . . .	205
9.3 Curves in the Complex Plane . . . . .	212
9.4 Key Points . . . . .	214
10. Dimensionality and Scaling	217
10.1 From Pythagoras to Einstein . . . . .	218
10.2 Simpler than how Niels Bohr did it . . . . .	220
10.3 The Bjerrum Length . . . . .	224
10.4 The Debye Length . . . . .	226
10.5 Dimensionless Constants . . . . .	228
10.6 Non-Universal Dimensionless Constants and their Usefulness . . . . .	230
10.7 Examples of Order of Magnitude Estimates and Deriving Relationships Based on Dimensional Analysis . . . . .	236

10.8	Back to Great Constants of Universe and Mathematics . . . . .	245
10.9	Key Points . . . . .	248
	<i>Concluding Remarks</i>	251
	<i>Problems</i>	253
 <b>Part II: From Camp 1: Deeper Understanding of Functions and Solving Equations</b>		<b>271</b>
1.	Introduction to Functions of Two and More Variables	273
1.1	Plotting Functions of Two Variables, Their Derivatives and Turning Points . . . . .	273
1.2	Changing Variables in the Generalized Chain Rule . . . . .	280
1.3	Small Changes . . . . .	285
1.4	Line Integrals along Paths . . . . .	289
1.5	Path Functions, State Functions, and Thermodynamics . . . . .	294
1.6	Surface and Volume Integration: A Quick Preview . . . . .	304
1.7	Key Points . . . . .	311
2.	Fourier Series and Integrals	313
2.1	Fourier Cosine Series . . . . .	313
2.2	Fourier Sine Series . . . . .	321
2.3	Mixed and Complex Fourier Series . . . . .	329
2.4	Fourier Transforms . . . . .	335
2.5	Multidimensional Fourier Series and Fourier Transforms . . . . .	346
2.6	Key Points . . . . .	358

3. Linear Equations and Determinants	361
3.1 Introduction to Linear Equations: Spectroscopy and Application to the Beer–Lambert Law . . . . .	361
3.2 Introducing Determinants and Cramer’s Rule for the Solution of Inhomogeneous Linear Equations . . . . .	365
3.3 Homogeneous Linear Equations . . . . .	382
3.4 Eigenvalue Equations: An Introduction Considering the $H_2^+$ Ion as an Example . . . . .	388
3.5 Key Points . . . . .	395
4. Matrices and Symmetry	397
4.1 Matrices and Matrix-Algebra . . . . .	397
4.2 Linear Equations Using Matrices and Their Diagonalization . . . . .	411
4.3 Matrices and Discrete Symmetry . . . . .	431
4.4 Matrix Coordinate Transformations and Coordinate Systems . . . . .	441
4.5 Key Points . . . . .	460
5. Solving Nonlinear Equations, Algebraic and Transcendental	463
5.1 Some Examples of the Solution of Nonlinear Algebraic and Transcendental Equations . . . . .	464
5.2 Solving Equations through Newton’s Method . . . . .	480
5.3 Lagrange Inversion . . . . .	490
5.4 Coupled Nonlinear Equations . . . . .	494
5.5 Key Points . . . . .	503
6. Introduction to Ordinary Differential Equations	507
6.1 First-Order Differential Equations . . . . .	507
6.2 Second-Order Linear Differential Equations with Constant Coefficients . . . . .	540

6.3	Reduction of Order . . . . .	562
6.4	Key Points . . . . .	579
7.	Further Methods for Evaluating Integrals and the Gamma Function	583
7.1	Methods for the Exact Evaluation of Integrals . . . .	583
7.2	Further Methods for the Approximation and Asymptotic Expansion of Integrals . . . . .	588
7.3	The Euler Gamma Function . . . . .	610
7.4	Key Points . . . . .	615
8.	Functions of a Complex Variable	617
8.1	Rudiments of Complex-Variable Functions . . . . .	618
8.2	Branch Cuts . . . . .	630
8.3	Key Points . . . . .	638
	<i>Concluding Remarks</i>	641
	<i>Problems</i>	645
	<i>Instructions to Access the Outlines of Solutions</i>	665