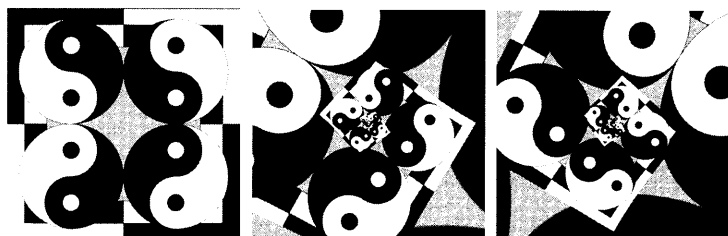


# Theory of Groups and Symmetries

**Finite Groups, Lie Groups, and Lie Algebras**



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