

# VARIATIONAL ANALYSIS IN SOBOLEV AND *BV* SPACES *Applications to PDEs and Optimization* SECOND EDITION

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