

**FRACTIONAL INTEGRALS
AND DERIVATIVES**

Theory and Applications

Stefan G. Samko

Rostov State University, Russia

Anatoly A. Kilbas

Belorussian State University, Minsk, Belarus

Oleg I. Marichev

Belorussian State University, Minsk, Belarus

Gordon and Breach Science Publishers

**Switzerland Australia Belgium France Germany Great Britain
India Japan Malaysia Netherlands Russia Singapore USA**

CONTENTS

Foreword	xv
Preface to the English edition	xvii
Preface	xix
Introduction	xxiii
Notation of the main forms of fractional integrals and derivatives	xxv
Brief historical outline	xxvii
Chapter 1 – Fractional Integrals and Derivatives on an Interval	1
§1. Preliminaries	1
1.1. The spaces H^λ and $H^\lambda(\rho)$	1
1.2. The spaces L_p and $L_p(\rho)$	7
1.3. Some special functions	14
1.4. Integral transforms	23
§2. Riemann-Liouville Fractional Integrals and Derivatives	28
2.1. The Abel integral equation	29
2.2. On the solvability of the Abel equation in the space of integrable functions	30
2.3. Definition of fractional integrals and derivatives and their simplest properties	33
2.4. Fractional integrals and derivatives of complex order	38
2.5. Fractional integrals of some elementary functions	40
2.6. Fractional integration and differentiation as reciprocal operations	43
2.7. Composition formulae. Connection with semigroups of operators	46

§3. The Fractional Integrals of Hölder and Summable Functions	53
3.1. Mapping properties in the space H^λ	53
3.2. Mapping properties in the space $H_0^\lambda(\rho)$	57
3.3. Mapping properties in the space L_p	66
3.4. Mapping properties in the space $L_p(\rho)$	70
§4. Bibliographical Remarks and Additional Information to Chapter 1	82
4.1. Historical notes	82
4.2. Survey of other results (relating to §§1-3)	84
Chapter 2 – Fractional Integrals and Derivatives on the Real Axis and Half-Axis	93
§5. The Main Properties of Fractional Integrals and Derivatives	93
5.1. Definitions and elementary properties	93
5.2. Fractional integrals of Hölderian functions	98
5.3. Fractional integrals of summable functions	102
5.4. The Marchaud fractional derivative	109
5.5. The finite part of integrals due to Hadamard	112
5.6. Properties of finite differences and Marchaud fractional derivatives of order $\alpha > 1$	116
5.7. Connection with fractional power of operators	120
§6. Representation of Function by Fractional Integrals of L_p-Functions	122
6.1. The space $I^\alpha(L_p)$	122
6.2. Inversion of fractional integrals of L_p -functions	123
6.3. Characterization of the space $I^\alpha(L_p)$	127
6.4. Sufficiency conditions for the representability of functions by fractional integrals	131
6.5. On the integral modulus of continuity of $I^\alpha(L_p)$ -functions ...	136
§7. Integral Transforms of Fractional Integrals and Derivatives	137
7.1. The Fourier transform	137
7.2. The Laplace transform	140
7.3. The Mellin transform	142
§8. Fractional Integrals and Derivatives of Generalized Functions	145
8.1. Preliminary ideas	145
8.2. The case of the axis R^1 . Lizorkin's space of test functions ...	146

8.3.	Schwartz's approach.....	154
8.4.	The case of the half-axis. The approach via the adjoint operator.....	155
8.5.	McBride's spaces.....	157
8.6.	The case of an interval.....	159
§9.	Bibliographical Remarks and Additional Information to Chapter 2.....	160
9.1.	Historical notes.....	160
9.2.	Survey of other results (relating to §§ 5-8).....	163
9.3.	Tables of fractional integrals and derivatives.....	172
Chapter 3 – Further Properties of Fractional Integrals and Derivatives.....		175
§10.	Compositions of Fractional Integrals and Derivatives with Weights.....	175
10.1.	Compositions of two one-sided integrals with power weights.....	176
10.2.	Compositions of two-sided integrals with power weights.....	189
10.3.	Compositions of several integrals with power weights.....	191
10.4.	Compositions with exponential and power-exponential weights.....	195
§11.	Connection between Fractional Integrals and the Singular Operator.....	199
11.1.	The singular operator S	199
11.2.	The case of the whole line.....	202
11.3.	The case of an interval and a half-axis.....	204
11.4.	Some other composition relations.....	210
§12.	Fractional Integrals of the Potential Type.....	213
12.1.	The real axis. The Riesz and Feller potentials.....	214
12.2.	On the "truncation" of the Riesz potential to the half-axis ..	218
12.3.	The case of the half-axis.....	221
12.4.	The case of a finite interval.....	222
§13.	Functions Representable by Fractional Integrals on an Interval.....	224
13.1.	The Marchaud fractional derivative on an interval.....	224
13.2.	Characterization of fractional integrals of functions in L_p ...	229
13.3.	Continuation, restriction and "sewing" of fractional integrals	234
13.4.	Characterization of fractional integrals of Hölderian functions	238

13.5.	Fractional integration in the union of weighted Hölder spaces	246
13.6.	Fractional integrals and derivatives of functions with a prescribed continuity modulus	249
§14.	Miscellaneous Results for Fractional Integro-differentiation of Functions of a Real Variable	254
14.1.	Lipschitz spaces H_p^λ and \tilde{H}_p^λ	254
14.2.	Mapping properties of fractional integration in H_p^λ	256
14.3.	Fractional integrals and derivatives of functions which are given on the whole line and belong to H_p^λ on every finite interval	261
14.4.	Fractional derivatives of absolutely continuous functions	267
14.5.	The Riesz mean value theorem and inequalities for fractional integrals and derivatives	270
14.6.	Fractional integration and the summation of series and integrals	275
§15.	The Generalized Leibniz Rule	277
15.1.	Fractional integro-differentiation of analytic functions on the real axis	277
15.2.	The generalized Leibniz rule	280
§16.	Asymptotic Expansions of Fractional Integrals	285
16.1.	Definitions and properties of asymptotic expansions	285
16.2.	The case of a power asymptotic expansion	287
16.3.	The case of a power-logarithmic asymptotic expansion	294
16.4.	The case of a power-exponential asymptotic expansion	297
16.5.	The asymptotic solution of Abel's equation	299
§17.	Bibliographical Remarks and Additional Information to Chapter 3	301
17.1.	Historical notes	301
17.2.	Survey of other results (relating to §§ 10-16)	305
Chapter 4	– Other Forms of Fractional Integrals and Derivatives	321
§18.	Direct Modifications and Generalizations of Riemann-Liouville Fractional Integrals	321
18.1.	Erdélyi-Kober-type operators	322
18.2.	Fractional integrals of a function by another function	325
18.3.	Hadamard fractional integro-differentiation	329

18.4.	One-dimensional modification of Bessel fractional integro-differentiation and the spaces $H^{s,p} = L_p^s$	333
18.5.	The Chen fractional integral	338
18.6.	Dzherbashyan's generalized fractional integral	344
§19.	Weyl Fractional Integrals and Derivatives of Periodic Functions	347
19.1.	Definitions. Connections with Fourier series	347
19.2.	Elementary properties of Weyl fractional integrals	352
19.3.	Other forms of fractional integration of periodic functions ...	354
19.4.	The coincidence of Weyl and Marchaud fractional derivatives	356
19.5.	The representability of periodic functions by the Weyl fractional integral	358
19.6.	Weyl fractional integration and differentiation in the space of Hölderian functions	361
19.7.	Weyl fractional integrals and derivatives of periodic functions in H_p^λ	367
19.8.	The Bernstein inequality for fractional integrals of trigonometric polynomials	368
§20.	An Approach to Fractional Integro-differentiation via Fractional Differences (The Grünwald-Letnikov Approach)	371
20.1.	Differences of a fractional order and their properties	371
20.2.	Coincidence of the Grünwald-Letnikov derivative with the Marchaud derivative. The periodic case	376
20.3.	Coincidence of the Grünwald-Letnikov derivative with the Marchaud derivative. The non-periodic case	382
20.4.	Grünwald-Letnikov fractional differentiation on a finite interval	385
§21.	Operators with Power-Logarithmic Kernels	388
21.1.	Mapping properties in the space H^λ	389
21.2.	Mapping properties in the space $H_0^\lambda(\rho)$	396
21.3.	Mapping properties in the space L_p	401
21.4.	Mapping properties in the space $L_p(\rho)$	404
21.5.	Asymptotic expansions	411
§22.	Fractional Integrals and Derivatives in the Complex Plane	414
22.1.	Definitions and the main properties of fractional integro-differentiation in the complex plane	416
22.2.	Fractional integro-differentiation of analytic functions	420
22.3.	Generalization of fractional integro-differentiation of analytic functions	426

§23. Bibliographical Remarks and Additional Information to Chapter 4	431
23.1. Historical notes	431
23.2. Survey of other results (relating to §§ 18-22)	436
23.3. Answers to some questions put at the Conference on Fractional Calculus (New Haven, 1974)	455
Chapter 5 – Fractional Integro-differentiation of Functions of Many Variables	457
§24. Partial and Mixed Integrals and Derivatives of Fractional Order	458
24.1. The multidimensional Abel integral equation	458
24.2. Partial and mixed fractional integrals and derivatives	459
24.3. The case of two variables. Tensor product of operators	463
24.4. Mapping properties of fractional integration operators in the spaces $L_{\bar{p}}(R^n)$ (with mixed norm)	464
24.5. Connection with a singular integral	466
24.6. Partial and mixed fractional derivatives in the Marchaud form	468
24.7. Characterization of fractional integrals of functions in $L_{\bar{p}}(R^2)$	471
24.8. Integral transform of fractional integrals and derivatives	473
24.9. Lizorkin function space invariant relative to fractional integro-differentiation	475
24.10. Fractional derivatives and integrals of periodic functions of many variables	476
24.11. Grünwald-Letnikov fractional differentiation	479
24.12. Operators of the polypotential type	480
§25. Riesz Fractional Integro-differentiation	483
25.1. Preliminaries	484
25.2. The Riesz potential and its Fourier transform. Invariant Lizorkin space	489
25.3. Mapping properties of the operator I^α in the spaces $L_p(R^n)$ and $L_p(R^n; \rho)$	494
25.4. Riesz differentiation (hypersingular integrals)	498
25.5. Unilateral Riesz potentials	502
§26. Hypersingular Integrals and the Space of Riesz Potentials	505
26.1. Investigation of the normalizing constants $d_{n,l}(\alpha)$ as functions of the parameter α	505

26.2.	Convergence of the hypersingular integral for smooth functions and diminution of order l to $l > 2[\alpha/2]$ in the case of a non-centered difference	510
26.3.	The hypersingular integral as an inverse of a Riesz potential	512
26.4.	Hypersingular integrals with homogeneous characteristics ...	518
26.5.	Hypersingular integral with a homogeneous characteristic as a convolution with the distribution	525
26.6.	Representation of differential operators in partial derivatives by hypersingular integrals	527
26.7.	The space $I^\alpha(L_p)$ of Riesz potentials and its characterization in terms of hypersingular integrals. The space $L_{p,r}^\alpha(R^n)$	532
§27.	Bessel Fractional Integro-differentiation	538
27.1.	The Bessel kernel and its properties	538
27.2.	Connections with Poisson, Gauss-Weierstrass and metaharmonic continuation semigroups.....	541
27.3.	The space of Bessel potentials	543
27.4.	The realization of $(E-\Delta)^{\alpha/2}$, $\alpha > 0$, in terms of hypersingular integrals	547
§28.	Other Forms of Multidimensional Fractional Integro-differentiation	554
28.1.	Riesz potential with Lorentz distance (hyperbolic Riesz potentials)	555
28.2.	Parabolic potentials	562
28.3.	The realization of the fractional powers $(-\Delta_x + \frac{\partial}{\partial t})^{\alpha/2}$ and $(E - \Delta_x + \frac{\partial}{\partial t})^{\alpha/2}$, $\alpha > 0$, in terms of a hypersingular integral	565
28.4.	Pyramidal analogues of mixed fractional integrals and derivatives	569
§29.	Bibliographical Remarks and Additional Information to Chapter 5.....	580
29.1.	Historical notes	580
29.2.	Survey of other results (relating to §§ 24-28).....	584
Chapter 6 – Applications to Integral Equations of the First Kind with Power and Power-Logarithmic Kernels		605
§30.	The Generalized Abel Integral Equation	606
30.1.	The dominant singular integral equation	606
30.2.	The generalized Abel equation on the whole axis.....	610
30.3.	The generalized Abel equation on an interval	616
30.4.	The case of constant coefficients	622

§31. The Noether Nature of the Equation of the First Kind with Power-Type Kernels	629
31.1. Preliminaries on Noether operators	630
31.2. The equation on the axis	634
31.3. Equations on a finite interval	646
31.4. On the stability of solutions	657
§32. Equations with Power-Logarithmic Kernels	659
32.1. Special Volterra functions and some of their properties	661
32.2. The solution of equations with integer non-negative powers of logarithms	664
32.3. The solution of equations with real powers of logarithms	667
§33. The Noether Nature of Equations of the First Kind with Power-Logarithmic Kernels	672
33.1. Imbedding theorems for the ranges of the operators $I_{a+}^{\alpha,\beta}$ and $I_{b-}^{\alpha,\beta}$	673
33.2. Connection between the operators with power-logarithmic kernels and singular operator	674
33.3. The Noether nature of equation (33.1)	681
§34. Bibliographical Remarks and Additional Information to Chapter 6	684
34.1. Historical notes	684
34.2. Survey of other results (relating to §§ 30-33)	687
Chapter 7 – Integral Equations of the First Kind with Special Functions as Kernels	695
§35. Some Equations with Homogeneous Kernels Involving Gauss and Legendre Functions	696
35.1. Equations with the Gauss function	696
35.2. Equations with the Legendre function	699
§36. Fractional Integrals and Derivatives as Integral Transforms	703
36.1. Definition of the G -transform. The spaces $\mathfrak{M}_{c,\gamma}^{-1}(L)$ and $L_2^{(c,\gamma)}$ and their characterization	704
36.2. Existence, mapping properties and representations of the G -transform:	709
36.3. Factorization of the G -transform	713
36.4. Inversion of the G -transform	716
36.5. The mapping properties, factorization and inversion of fractional integrals in the spaces $\mathfrak{M}_{c,\gamma}^{-1}(L)$ and $L_2^{(c,\gamma)}$	720

36.6. Other examples of factorization.....	722
36.7. Mapping properties of the G -transform on fractional integrals and derivatives.....	726
36.8. Index laws for fractional integrals and derivatives.....	727
§37. Equations with Non-Homogeneous Kernels.....	730
37.1. Equations with difference kernels.....	731
37.2. Generalized operators of Hankel and Erdélyi-Kober transforms.....	737
37.3. Non-convolution operators with Bessel functions in kernels..	741
37.4. Equation of compositional type.....	746
37.5. The W -transform and its inversion.....	752
37.6. Application of fractional integrals to the inversion of the W -transform.....	758
§38. Applications of Fractional Integro-differentiation to the Investigation of Dual Integral Equations.....	761
38.1. Dual Equations.....	762
38.2. Triple equations.....	768
§39. Bibliographical Remarks and Additional Information to Chapter 7.....	772
39.1. Historical notes.....	772
39.2. Survey of other results (relating to §§ 35-38).....	775
Chapter 8 – Applications to Differential Equations.....	795
§40. Integral Representations for Solution of Partial Differential Equations of the Second Order via Analytic Functions and Their Applications to Boundary Value Problems.....	795
40.1. Preliminaries.....	796
40.2. The representation of solutions of generalized Helmholtz two-axially symmetric equation.....	800
40.3. Boundary value problems for the generalized Helmholtz two-axially symmetric equation.....	809
§41. Euler-Poisson-Darboux Equation.....	812
41.1. Representations for solutions of the Euler-Poisson-Darboux equation.....	813
41.2. Classical and generalized solutions of the Cauchy problem..	819
41.3. The half-homogeneous Cauchy problem in multidimensional half-space.....	823
41.4. The weighted Dirichlet and Neumann problems in a half-plane	826

§42. Ordinary Differential Equations of Fractional Order	829
42.1. The Cauchy-type problem for differential equations and systems of differential equations of fractional order of general form	830
42.2. The Cauchy-type problem for linear differential equation of fractional order	837
42.3. The Dirichlet-type problem for linear differential equation of fractional order	843
42.4. Solution of the linear differential equation of fractional order with constant coefficients in the space of generalized functions	846
42.5. The application of fractional differentiation to differential equations of integer order	849
§43. Bibliographical Remarks and Additional Information to Chapter 8	856
43.1. Historical notes	856
43.2. Survey of other results (relating to §§ 40-42)	858
Bibliography	873
Author Index	953
Subject Index	965
Index of Symbols	973