

Henning Struchtrup

Macroscopic Transport Equations for Rarefied Gas Flows

Approximation Methods in Kinetic Theory

With 35 Figures

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