

# Contents

SERIES EDITOR'S PREFACE	v
PREFACE	xi
<b>PART I. PD-Operators with Complex Arguments</b>	<b>1</b>
Introduction	1
<b>CHAPTER 1. PD-Operators with Constant Analytic Symbols</b>	<b>5</b>
1.1. Spaces of entire functions of exponential type	5
1. The space $\text{Exp}_{\mathbb{R}}(\mathbb{C}_z^n)$ (5). 2. Estimates of derivatives (7). 3. The test-function space $\text{Exp}_{\Omega}(\mathbb{C}_z^n)$ . Topology and convergence (8). 4. Density of linear combinations of $\exp \lambda z$ , $\lambda \in \Omega$ (12).	
1.2. PD-operators with analytic symbols	15
1. Local algebra of differential operators of infinite order (15). 2. Algebra of PD-operators with arbitrary analytic symbols (20). 3. The correctness of the definition of a PD-operator (23).	
1.3. The operator method	28
1. PD-equations in the whole space $\mathbb{C}^n$ (28). 2. The Cauchy problem in the space of exponential functions (30). 3. Cauchy-Kovalevskaya theorem (special case) (35). 4. A two-point boundary value problem (40).	
1.4. The dual theory	41
1. Exponential functionals. Examples (41). 2. The general form of exponential functionals (43). 3. The algebra of PD-operators in the space of exponential functionals (45). 4. Cauchy problem in exponential functionals (47).	
<b>CHAPTER 2. Fourier Transformation of Arbitrary Analytic Functions. Complex Fourier Method</b>	<b>52</b>
2.1. Fourier transformation	52
1. Main definition. The inversion formula (52). 2. The Fourier image of exponential functions. The Borel kernel (54). 3. Complex unitarity (58).	

2.2. Complex Fourier method . . . . .	59
1. Table of duality. Examples (59). 2. Fourier method for PD-equations (62).	
<b>CHAPTER 3. PD-Equations whose Symbols are Formal Series . . . . .</b>	<b>66</b>
3.1. Differential operators of infinite order with constant coefficients . . . . .	66
1. The space $E_{q,r}(\mathbb{C}_z^n)$ of entire functions of order $q < 1$ (66). 2. The basic estimate (67). 3. The non-formal action of d.o.i.o.'s (70). 4. The algebra of d.o.i.o.'s (72). 5. The Cauchy problem (73).	
3.2. Differential operators of infinite order with variable coefficients . . . . .	76
1. Definition of a d.o.i.o. with variable coefficients (76). 2. The Cauchy problem in the spaces $E_{q,r}(\mathbb{C}_z^n)$ (78). 3. The Cauchy problem in the spaces $E_{q,r+\sigma t }(\mathbb{C}_z^n)$ (80).	
<b>PART II. The Cauchy Problem in the Complex Domain . . . . .</b>	<b>85</b>
Introduction . . . . .	85
<b>CHAPTER 4. Cauchy-Kovalevskaya Theory in Spaces of Analytic Functions with Pole-type Singularities . . . . .</b>	<b>90</b>
4.1. The Cauchy problem in the spaces $D_{m,r}$ (Case of cylindrical evolution) . . . . .	90
1. The test-function space $D_{m,r}$ (90). 2. Criterion for the well-posedness of the Cauchy problem in the spaces $D_{m,r}$ (91). 3. The structure of systems with $\text{ord}A_{ij} \leq m_i - m_j$ (96). 4. The Cauchy problem in the dual spaces $D'_{m,r}$ (98).	
4.2. The Cauchy problem in the spaces $D_{m,r-\sigma t }$ . (Case of conical evolution) . . . . .	101
1. The test-function space $D_{m,r-\sigma t }$ (101). 2. Criterion for the well-posedness of the Cauchy problem in the spaces $D_{m,r-\sigma t }$ . Necessity of Kovalevskaya conditions and Leray-Volevich conditions (102). 3. The structure of systems with $\text{ord}A_{ij} \leq m_i - m_j + 1$ (106). 4. The Cauchy problem in the dual scale $D'_{m+1,r+\sigma t }$ (110).	
4.3. Formulation of the basic results for arbitrary systems in normal form	117
<b>CHAPTER 5. Exponential theory of the Cauchy problem . . . . .</b>	<b>120</b>
5.1. The Cauchy problem in the scale of spaces of initial data . . . . .	120
1. Banach spaces of entire functions of finite order (120). 2. Criterion for the well-posedness of the Cauchy problem in the scale	

$\text{Exp}_{m,r,q}(\mathbf{C}_z^n)$  (123). 3. The structure of systems satisfying the conditions  $\deg a_{ij}^\alpha \leq m_i - m_j - |\alpha|(q - 1)$  (129).

5.2. The Cauchy problem in the scale of “linearly increasing” spaces of initial data . . . . . 133

1. The scale  $\text{Exp}_{m,r+\sigma|t|,q}(\mathbf{C}_z^n)$  of entire functions of finite order (133).  
 2. Criterion for the well-posedness of the Cauchy problem in the scale  $\text{Exp}_{m,r+\sigma|t|,q}(\mathbf{C}_z^n)$  (134). 3. The structure of systems with  $\deg a_{ij}^\alpha \leq m_i - m_j - |\alpha|(q - 1) + q$  (141). 4. The Cauchy problem in the dual space  $\text{Exp}'_{m,r-\sigma|t|,q}(\mathbf{C}_z^n)$  (151).

CHAPTER 6. PD-Operators with Variable Analytic Symbols . . . 157

6.1. Basic definitions . . . . . 157

1. Definition of the PD-operator  $A(z, \mathcal{D})$  (157). 2. Definitions of the requisite spaces (159).

6.2. The Cauchy problem in the scale  $\text{Exp}_{m,r}(\mathbf{C}_z^n)$  . . . . . 160

6.3. The Cauchy problem in the scale  $\text{Exp}_{m,r+\sigma|t|}(\mathbf{C}_z^n)$  . . . . . 166

Conclusion: The connection between the Cauchy exponential theory and the classical Cauchy-Kovalevskaya theory . . . . . 171

PART III. PD-Operators with Real Arguments . . . . . 174

Introduction . . . . . 174

CHAPTER 7. Spaces of Test Functions and Distributions . . . . . 176

7.1. The test-function space  $H^\infty(S_R)$  . . . . . 176

1. Definition and examples of test functions (176). 2. Description of  $H^\infty(S_R)$  in the  $x$ -variables (178). 3. Convergence in  $H^\infty(S_R)$  (179).  
 4. Invariance of  $H^\infty(S_R)$  under differential operators of infinite order (180). 5. An example (181).

7.2. The generalized function space  $H^{-\infty}(S_R)$  . . . . . 182

1. Definition of  $H^{-\infty}(S_R)$ . Main property (182). 2. Examples of functionals in  $H^{-\infty}(S_R)$  (185).

7.3. Sobolev spaces of infinite order  $W^\infty\{a_\alpha, p\}(\mathbf{R}^n)$  . . . . . 187

1. Criterion for non-triviality of the spaces  $W^\infty\{a_\alpha, p\}(\mathbf{R}^n)$  (188).  
 2. The distribution space  $W^{-\infty}\{a_\alpha, p'\}(\mathbf{R}^n)$  (193).

CHAPTER 8. Analytic PD-operators with Real Arguments.  
 Applications . . . . . 196

8.1. Algebra of PD-operators with analytic symbols . . . . . 196

1. The space $H^\infty(G)$ (196). 2. The action of PD-operators (198). 3. Examples (201). 4. The dual theory (203). 5. A possible generalization (205).	
8.2. PD-equations in the whole Euclidean space . . . . .	206
8.3. The Cauchy problem . . . . .	208
1. The Cauchy problem in the space $H^\infty(G)$ (208). 2. The Cauchy problem in the dual space $H^{-\infty}(G)$ (211). 3. On the existence of the fundamental solution of the Cauchy problem (213).	
8.4. Examples . . . . .	214
1. The Cauchy problem for a homogeneous equation (214). 2. Special case. Laplace equation (216). 3. The Cauchy problem for the heat equation (218). 4. One equation with a shift (218). 5. Quasi-polynomial solutions (219). 6. One boundary value problem in a strip (220). 7. The boundary value problem in the cylinder (221). 8. The Dirichlet problem in a disc. Poisson integral (222). 9. The Dirichlet problem in the half-plane. Cauchy integral (224). 10. Analytic continuation of a pair of functions defined on $\mathbf{R}^1$ (225).	
8.5. Quantum relativistic particle with zero spin . . . . .	226
1. Derivation of the Schrödinger equation (226). 2. Fundamental solution of the Cauchy problem (230). 3. Lorentz invariance (233). 4. Description of Lorentz-invariant solutions (236). 5. Recurrence formulae for the Lorentz-invariant solutions (238). 6. Non-relativistic limit and factorization of Klein-Gordon-Fock operator (239).	
REFERENCES . . . . .	242
AUTHOR INDEX . . . . .	248
INDEX OF BASIC FORMULAE . . . . .	250
SUBJECT INDEX . . . . .	251