

A FIRST COURSE IN FUNCTIONAL ANALYSIS

S. DAVID PROMISLOW

Department of Mathematics and Statistics

York University

Toronto, Ontario, Canada

CONTENTS

Preface	xi
1. Linear Spaces and Operators	1
1.1 Introduction	1
1.2 Linear Spaces	2
1.3 Linear Operators	5
1.4 Passage from Finite- to Infinite-Dimensional Spaces	7
Exercises	8
2. Normed Linear Spaces: The Basics	11
2.1 Metric Spaces	11
2.2 Norms	12
2.3 Space of Bounded Functions	18
2.4 Bounded Linear Operators	19
2.5 Completeness	21
2.6 Comparison of Norms	30
2.7 Quotient Spaces	31
2.8 Finite-Dimensional Normed Linear Spaces	34
2.9 L^p Spaces	38
2.10 Direct Products and Sums	51
2.11 Schauder Bases	53
2.12 Fixed Points and Contraction Mappings	53
Exercises	54
3. Major Banach Space Theorems	59
3.1 Introduction	59
3.2 Baire Category Theorem	59
3.3 Open Mappings	61
3.4 Bounded Inverses	63

3.5	Closed Linear Operators	64
3.6	Uniform Boundedness Principle	66
	Exercises	68
4.	Hilbert Spaces	71
4.1	Introduction	71
4.2	Semi-Inner Products	72
4.3	Nearest Points and Convexity	77
4.4	Orthogonality	80
4.5	Linear Functionals on Hilbert Spaces	86
4.6	Linear Operators on Hilbert Spaces	88
4.7	Order Relation on Self-Adjoint Operators	97
	Exercises	98
5.	Hahn–Banach Theorem	103
5.1	Introduction	103
5.2	Basic Version of Hahn–Banach Theorem	104
5.3	Complex Version of Hahn–Banach Theorem	105
5.4	Application to Normed Linear Spaces	107
5.5	Geometric Versions of Hahn–Banach Theorem	108
	Exercises	118
6.	Duality	121
6.1	Examples of Dual Spaces	121
6.2	Adjoints	130
6.3	Double Duals and Reflexivity	133
6.4	Weak and Weak* Convergence	136
	Exercises	140
7.	Topological Linear Spaces	143
7.1	Review of General Topology	143
7.2	Topologies on Linear Spaces	148
7.3	Linear Functionals on Topological Linear Spaces	151
7.4	Weak Topology	153
7.5	Weak* Topology	156
7.6	Extreme Points and Krein–Milman Theorem	160
7.7	Operator Topologies	164
	Exercises	164
8.	The Spectrum	167
8.1	Introduction	167
8.2	Banach Algebras	169

8.3	General Properties of the Spectrum	170
8.4	Numerical Range	176
8.5	Spectrum of a Normal Operator	177
8.6	Functions of Operators	180
8.7	Brief Introduction to C^* -Algebras	183
	Exercises	184
9.	Compact Operators	187
9.1	Introduction and Basic Definitions	187
9.2	Compactness Criteria in Metric Spaces	188
9.3	New Compact Operators from Old	192
9.4	Spectrum of a Compact Operator	194
9.5	Compact Self-Adjoint Operators on Hilbert Spaces	197
9.6	Invariant Subspaces	201
	Exercises	203
10.	Application to Integral and Differential Equations	205
10.1	Introduction	205
10.2	Integral Operators	206
10.3	Integral Equations	211
10.4	Second-Order Linear Differential Equations	214
10.5	Sturm–Liouville Problems	217
10.6	First-Order Differential Equations	223
	Exercises	226
11.	Spectral Theorem for Bounded, Self-Adjoint Operators	229
11.1	Introduction and Motivation	229
11.2	Spectral Decomposition	231
11.3	Extension of Functional Calculus	235
11.4	Multiplication Operators	240
	Exercises	243
Appendix A	Zorn’s Lemma	245
Appendix B	Stone–Weierstrass Theorem	247
B.1	Basic Theorem	247
B.2	Nonunital Algebras	250
B.3	Complex Algebras	252
Appendix C	Extended Real Numbers and Limit Points of Sequences	253
C.1	Extended Reals	253
C.2	Limit Points of Sequences	254

Appendix D	Measure and Integration	257
D.1	Introduction and Notation	257
D.2	Basic Properties of Measures	258
D.3	Properties of Measurable Functions	259
D.4	Integral of a Nonnegative Function	261
D.5	Integral of an Extended Real-Valued Function	265
D.6	Integral of a Complex-Valued Function	267
D.7	Construction of Lebesgue Measure on \mathcal{R}	267
D.8	Completeness of Measures	273
D.9	Signed and Complex Measures	274
D.10	Radon–Nikodym Derivatives	276
D.11	Product Measures	278
D.12	Riesz Representation Theorem	280
Appendix E	Tychonoff's Theorem	289
Symbols		293
References		297
Index		299