

Zariski-decomposition and Abundance

Noboru Nakayama

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES,
KYOTO UNIVERSITY, KYOTO 606-8502 JAPAN
E-mail address: nakayama@kurims.kyoto-u.ac.jp

Dedicated to Professor Shigeru Itaka on the occasion of his sixtieth birthday

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