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Phase Transitions in Combinatorial Optimization Problems

Basics, Algorithms and Statistical Mechanics



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