

David E. Blair

Riemannian Geometry of Contact and Symplectic Manifolds

Birkhäuser
Boston • Basel • Berlin

Contents

1	Symplectic Manifolds	1
1.1	Definitions and examples	1
1.2	Lagrangian submanifolds	5
1.3	The Darboux–Weinstein theorems	7
1.4	Symplectomorphisms	9
2	Principal S^1-bundles	11
2.1	The set of principal S^1 -bundles as a group	11
2.2	Connections on a principal bundle	14
3	Contact Manifolds	17
3.1	Definitions	17
3.2	Examples	20
3.2.1	\mathbb{R}^{2n+1}	20
3.2.2	$\mathbb{R}^{n+1} \times P\mathbb{R}^n$	21
3.2.3	$M^{2n+1} \subset \mathbb{R}^{2n+2}$ with $T_m M^{2n+1} \cap \{0\} = \emptyset$	21
3.2.4	T_1^*M, T_1M	22
3.2.5	$T^*M \times \mathbb{R}$	22
3.2.6	T^3	23
3.2.7	T^5	23
3.2.8	Overtwisted contact structures	24
3.2.9	Contact circles	25
3.3	The Boothby–Wang fibration	26
3.4	The Weinstein conjecture	28

4 Associated Metrics	31
4.1 Almost complex and almost contact structures	31
4.2 Polarization and associated metrics	34
4.3 Polarization of metrics as a projection	38
4.3.1 Some linear algebra	39
4.3.2 Results on the set \mathcal{A}	41
4.4 Action of symplectic and contact transformations	45
4.5 Examples of almost contact metric manifolds	48
4.5.1 \mathbb{R}^{2n+1}	48
4.5.2 $M^{2n+1} \subset \tilde{M}^{2n+2}$ almost complex	49
4.5.3 $S^5 \subset S^6$	50
4.5.4 The Boothby–Wang fibration	52
4.5.5 $M^{2n} \times \mathbb{R}$	53
4.5.6 Parallelizable manifolds	53
5 Integral Submanifolds and Contact Transformations	55
5.1 Integral submanifolds	55
5.2 Contact transformations	57
5.3 Examples of integral submanifolds	59
5.3.1 $S^n \subset S^{2n+1}$	59
5.3.2 $T^2 \subset S^5$	59
5.3.3 Legendre curves and Whitney spheres	60
5.3.4 Lift of a Lagrangian submanifold	62
6 Sasakian and Cosymplectic Manifolds	63
6.1 Normal almost contact structures	63
6.2 The tensor field h	67
6.3 Definition of a Sasakian manifold	69
6.4 CR-manifolds	72
6.5 Cosymplectic manifolds and remarks on the Sasakian definition	77
6.6 Products of almost contact manifolds	79
6.7 Examples	81
6.7.1 \mathbb{R}^{2n+1}	81
6.7.2 Principal circle bundles	81
6.7.3 A non-normal almost contact structure on S^5	83
6.7.4 $M^{2n+1} \subset \tilde{M}^{2n+2}$	85
6.7.5 Brieskorn manifolds	85
6.8 Topology	87
7 Curvature of Contact Metric Manifolds	91
7.1 Basic curvature properties	91
7.2 Curvature of contact metric manifolds	95

7.3	ϕ -sectional curvature	110
7.4	Examples of Sasakian space forms	114
7.4.1	S^{2n+1}	114
7.4.2	\mathbb{R}^{2n+1}	114
7.4.3	$B^n \times \mathbb{R}$	115
7.5	Locally ϕ -symmetric spaces	115
8	Submanifolds of Kähler and Sasakian Manifolds	121
8.1	Invariant submanifolds	121
8.2	Lagrangian and integral submanifolds	124
8.3	Legendre curves	133
9	Tangent Bundles and Tangent Sphere Bundles	137
9.1	Tangent bundles	137
9.2	Tangent sphere bundles	142
9.3	Geometry of vector bundles	148
9.4	Normal bundles	150
10	Curvature Functionals on Spaces of Associated Metrics	157
10.1	Introduction to critical metric problems	157
10.2	The $*$ -scalar curvature	162
10.3	The integral of $Ric(\xi)$	166
10.4	The Webster scalar curvature	170
10.5	A gauge invariant	173
10.6	The Abbena metric as a critical point	174
11	Negative ξ-sectional Curvature	177
11.1	Special Directions in the contact subbundle	177
11.2	Anosov flows	178
11.3	Conformally Anosov flows	184
12	Complex Contact Manifolds	189
12.1	Complex contact manifolds and associated metrics	189
12.2	Examples of complex contact manifolds	193
12.2.1	Complex Heisenberg group	193
12.2.2	Odd-dimensional complex projective space	194
12.2.3	Twistor spaces	196
12.2.4	The complex Boothby–Wang fibration	198
12.2.5	3-dimensional homogeneous examples	200
12.2.6	$\mathbb{C}^{n+1} \times \mathbb{C}P^n(16)$	200
12.3	Normality of complex contact manifolds	202
12.4	GH -sectional curvature	204

x Contents

12.5 The set of associated metrics and integral functionals	206
12.6 Holomorphic Legendre curves	209
12.7 The Calabi (Veronese) imbeddings as integral submanifolds of $\mathbb{C}P^{2n+1}$	212
13 3-Sasakian Manifolds	215
13.1 3-Sasakian manifolds	215
13.2 Integral submanifolds	223
Bibliography	227
Subject Index	253
Author Index	257