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# Mathematical Models



*Mechanical Vibrations, Population Dynamics,  
and Traffic Flow*

An Introduction to Applied Mathematics

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(Starred sections may be  
omitted without loss of continuity.)

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