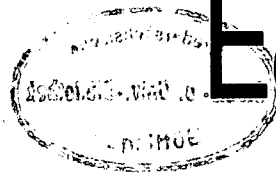


# Computer Methods for Ordinary Differential Equations and Differential–Algebraic Equations



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