

# MATHEMATICAL

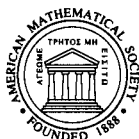
## Surveys and Monographs

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Volume 43

# Conjugacy Classes in Semisimple Algebraic Groups

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