Flag-signs in the margins designate a route: either Route 1 (which is entirely contained in Route 2), or Route 2 (which is entirely contained in Route 3), or Route 3; see the Preface and Introduction for a more detailed description of these routes. A new flag-sign is posted only at the moment of a route change.

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