The SABR/LIBOR Market Model
Pricing, Calibration and Hedging for Complex Interest-Rate Derivatives

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