

GRADUATE STUDIES **174**
IN MATHEMATICS

Quiver Representations and Quiver Varieties

Alexander Kirillov Jr.



American Mathematical Society
Providence, Rhode Island

Contents

| | |
|---|----|
| Preface | xi |
| Part 1. Dynkin Quivers | |
| Chapter 1. Basic Theory | 3 |
| §1.1. Basic definitions | 3 |
| §1.2. Path algebra; simple and indecomposable representations | 7 |
| §1.3. K -group and dimension | 11 |
| §1.4. Projective modules and the standard resolution | 11 |
| §1.5. Euler form | 15 |
| §1.6. Dynkin and Euclidean graphs | 16 |
| §1.7. Root lattice and Weyl group | 20 |
| Chapter 2. Geometry of Orbits | 23 |
| §2.1. Representation space | 23 |
| §2.2. Properties of orbits | 24 |
| §2.3. Closed orbits | 26 |
| Chapter 3. Gabriel's Theorem | 31 |
| §3.1. Quivers of finite type | 31 |
| §3.2. Reflection functors | 32 |
| §3.3. Dynkin quivers | 38 |
| §3.4. Coxeter element | 41 |
| §3.5. Longest element and ordering of positive roots | 43 |

| | |
|--|-----|
| Chapter 4. Hall Algebras | 47 |
| §4.1. Definition of Hall algebra | 47 |
| §4.2. Serre relations and Ringel's theorem | 52 |
| §4.3. PBW basis | 56 |
| §4.4. Hall algebra of constructible functions | 61 |
| §4.5. Finite fields vs. complex numbers | 66 |
| Chapter 5. Double Quivers | 69 |
| §5.1. The double quiver | 69 |
| §5.2. Preprojective algebra | 70 |
| §5.3. Varieties $\Lambda(\mathbf{v})$ | 72 |
| §5.4. Composition algebra of the double quiver | 75 |
| Part 2. Quivers of Infinite Type | |
| Chapter 6. Coxeter Functor and Preprojective Representations | 83 |
| §6.1. Coxeter functor | 84 |
| §6.2. Preprojective and preinjective representations | 86 |
| §6.3. Auslander–Reiten quiver: Combinatorics | 88 |
| §6.4. Auslander–Reiten quiver: Representation theory | 92 |
| §6.5. Preprojective algebra and Auslander–Reiten quiver | 96 |
| Chapter 7. Tame and Wild Quivers | 103 |
| §7.1. Tame-wild dichotomy | 103 |
| §7.2. Representations of the cyclic quiver | 105 |
| §7.3. Affine root systems | 106 |
| §7.4. Affine Coxeter element | 107 |
| §7.5. Preprojective, preinjective, and regular representations | 112 |
| §7.6. Category of regular representations | 113 |
| §7.7. Representations of the Kronecker quiver | 118 |
| §7.8. Classification of regular representations | 121 |
| §7.9. Euclidean quivers are tame | 126 |
| §7.10. Non-Euclidean quivers are wild | 127 |
| §7.11. Kac's theorem | 129 |

| | |
|---|-----|
| Chapter 8. McKay Correspondence and Representations of Euclidean Quivers | 133 |
| §8.1. Finite subgroups in $SU(2)$ and regular polyhedra | 133 |
| §8.2. ADE classification of finite subgroups | 135 |
| §8.3. McKay correspondence | 141 |
| §8.4. Geometric construction of representations of Euclidean quivers | 146 |
| Part 3. Quiver Varieties | |
| Chapter 9. Hamiltonian Reduction and Geometric Invariant Theory | 159 |
| §9.1. Quotient spaces in differential geometry | 159 |
| §9.2. Overview of geometric invariant theory | 160 |
| §9.3. Relative invariants | 163 |
| §9.4. Regular points and resolution of singularities | 168 |
| §9.5. Basic definitions of symplectic geometry | 171 |
| §9.6. Hamiltonian actions and moment map | 174 |
| §9.7. Hamiltonian reduction | 177 |
| §9.8. Symplectic resolution of singularities and Springer resolution | 180 |
| §9.9. Kähler quotients | 182 |
| §9.10. Hyperkähler quotients | 186 |
| Chapter 10. Quiver Varieties | 191 |
| §10.1. GIT quotients for quiver representations | 191 |
| §10.2. GIT moduli spaces for double quivers | 195 |
| §10.3. Framed representations | 200 |
| §10.4. Framed representations of double quivers | 204 |
| §10.5. Stability conditions | 206 |
| §10.6. Quiver varieties as symplectic resolutions | 210 |
| §10.7. Example: Type A quivers and flag varieties | 212 |
| §10.8. Hyperkähler construction of quiver varieties | 216 |
| §10.9. \mathbb{C}^\times action and exceptional fiber | 219 |

| | |
|---|-----|
| Chapter 11. Jordan Quiver and Hilbert Schemes | 225 |
| §11.1. Hilbert schemes | 225 |
| §11.2. Quiver varieties for the Jordan quiver | 227 |
| §11.3. Moduli space of torsion free sheaves | 230 |
| §11.4. Anti-self-dual connections | 235 |
| §11.5. Instantons on \mathbb{R}^4 and ADHM construction | 238 |
| Chapter 12. Kleinian Singularities and Geometric McKay Correspondence | 241 |
| §12.1. Kleinian singularities | 241 |
| §12.2. Resolution of Kleinian singularities via Hilbert schemes | 243 |
| §12.3. Quiver varieties as resolutions of Kleinian singularities | 245 |
| §12.4. Exceptional fiber and geometric McKay correspondence | 248 |
| §12.5. Instantons on ALE spaces | 253 |
| Chapter 13. Geometric Realization of Kac–Moody Lie Algebras | 259 |
| §13.1. Borel–Moore homology | 259 |
| §13.2. Convolution algebras | 261 |
| §13.3. Steinberg varieties | 264 |
| §13.4. Geometric realization of Kac–Moody Lie algebras | 266 |
| Appendix A. Kac–Moody Algebras and Weyl Groups | 273 |
| §A.1. Cartan matrices and root lattices | 273 |
| §A.2. Weight lattice | 274 |
| §A.3. Bilinear form and classification of Cartan matrices | 275 |
| §A.4. Weyl group | 276 |
| §A.5. Kac–Moody algebra | 277 |
| §A.6. Root system | 278 |
| §A.7. Reduced expressions | 280 |
| §A.8. Universal enveloping algebra | 281 |
| §A.9. Representations of Kac–Moody algebras | 282 |
| Bibliography | 285 |
| Index | 293 |