Linear Algebra and Matrix Analysis for Statistics

Sudipto Banerjee
Professor of Biostatistics
School of Public Health
University of Minnesota, U.S.A.

Anindya Roy
Professor of Statistics
Department of Mathematics and Statistics
University of Maryland, Baltimore County, U.S.A.
## Contents

**Preface** \(\text{ xv}\)

### 1 Matrices, Vectors and Their Operations \(\text{ 1}\)
- 1.1 Basic definitions and notations \(\text{ 2}\)
- 1.2 Matrix addition and scalar-matrix multiplication \(\text{ 5}\)
- 1.3 Matrix multiplication \(\text{ 7}\)
- 1.4 Partitioned matrices \(\text{ 14}\)
  - 1.4.1 \(2 \times 2\) partitioned matrices \(\text{ 14}\)
  - 1.4.2 General partitioned matrices \(\text{ 16}\)
- 1.5 The “trace” of a square matrix \(\text{ 18}\)
- 1.6 Some special matrices \(\text{ 20}\)
  - 1.6.1 Permutation matrices \(\text{ 20}\)
  - 1.6.2 Triangular matrices \(\text{ 22}\)
  - 1.6.3 Hessenberg matrices \(\text{ 24}\)
  - 1.6.4 Sparse matrices \(\text{ 26}\)
  - 1.6.5 Banded matrices \(\text{ 27}\)
- 1.7 Exercises \(\text{ 29}\)

### 2 Systems of Linear Equations \(\text{ 33}\)
- 2.1 Introduction \(\text{ 33}\)
- 2.2 Gaussian elimination \(\text{ 34}\)
- 2.3 Gauss-Jordan elimination \(\text{ 42}\)
- 2.4 Elementary matrices \(\text{ 44}\)
- 2.5 Homogeneous linear systems \(\text{ 48}\)
- 2.6 The inverse of a matrix \(\text{ 51}\)
- 2.7 Exercises \(\text{ 61}\)
9 Revisiting Linear Equations
  9.1 Introduction
  9.2 Null spaces and the general solution of linear systems
  9.3 Rank and linear systems
  9.4 Generalized inverse of a matrix
  9.5 Generalized inverses and linear systems
  9.6 The Moore-Penrose inverse
  9.7 Exercises

10 Determinants
  10.1 Introduction
  10.2 Some basic properties of determinants
  10.3 Determinant of products
  10.4 Computing determinants
  10.5 The determinant of the transpose of a matrix—revisited
  10.6 Determinants of partitioned matrices
  10.7 Cofactors and expansion theorems
  10.8 The minor and the rank of a matrix
  10.9 The Cauchy-Binet formula
  10.10 The Laplace expansion
  10.11 Exercises

11 Eigenvalues and Eigenvectors
  11.1 The Eigenvalue equation
  11.2 Characteristic polynomial and its roots
  11.3 Eigenspaces and multiplicities
  11.4 Diagonalizable matrices
  11.5 Similarity with triangular matrices
  11.6 Matrix polynomials and the Caley-Hamilton Theorem
  11.7 Spectral decomposition of real symmetric matrices
  11.8 Computation of eigenvalues
  11.9 Exercises
 CONTENTS

12 Singular Value and Jordan Decompositions 371
   12.1 Singular value decomposition 371
   12.2 The SVD and the four fundamental subspaces 379
   12.3 SVD and linear systems 381
   12.4 SVD, data compression and principal components 383
   12.5 Computing the SVD 385
   12.6 The Jordan Canonical Form 389
   12.7 Implications of the Jordan Canonical Form 397
   12.8 Exercises 399

13 Quadratic Forms 401
   13.1 Introduction 401
   13.2 Quadratic forms 402
   13.3 Matrices in quadratic forms 405
   13.4 Positive and nonnegative definite matrices 411
   13.5 Congruence and Sylvester's Law of Inertia 419
   13.6 Nonnegative definite matrices and minors 423
   13.7 Some inequalities related to quadratic forms 425
   13.8 Simultaneous diagonalization and the generalized eigenvalue problem 434
   13.9 Exercises 441

14 The Kronecker Product and Related Operations 445
   14.1 Bilinear interpolation and the Kronecker product 445
   14.2 Basic properties of Kronecker products 446
   14.3 Inverses, rank and nonsingularity of Kronecker products 453
   14.4 Matrix factorizations for Kronecker products 455
   14.5 Eigenvalues and determinant 460
   14.6 The vec and commutator operators 461
   14.7 Linear systems involving Kronecker products 466
   14.8 Sylvester's equation and the Kronecker sum 470
   14.9 The Hadamard product 472
   14.10 Exercises 480
15 Linear Iterative Systems, Norms and Convergence

15.1 Linear iterative systems and convergence of matrix powers 483
15.2 Vector norms 485
15.3 Spectral radius and matrix convergence 489
15.4 Matrix norms and the Gerschgorin circles 491
15.5 The singular value decomposition—revisited 499
15.6 Web page ranking and Markov chains 503
15.7 Iterative algorithms for solving linear equations 511
  15.7.1 The Jacobi method 512
  15.7.2 The Gauss-Seidel method 513
  15.7.3 The Successive Over-Relaxation (SOR) method 514
  15.7.4 The conjugate gradient method 514
15.8 Exercises 517

16 Abstract Linear Algebra

16.1 General vector spaces 521
16.2 General inner products 528
16.3 Linear transformations, adjoint and rank 531
16.4 The four fundamental subspaces—revisited 535
16.5 Inverses of linear transformations 537
16.6 Linear transformations and matrices 540
16.7 Change of bases, equivalence and similar matrices 543
16.8 Hilbert spaces 547
16.9 Exercises 552

References 555

Index 559