Vladimir Maz'ya • Alexander Movchan
Michael Nieves

Green's Kernels and Meso-Scale Approximations in Perforated Domains

Springer

Contents

Part I Green's Functions in Singularly Perturbed Domains

1 Uniform Asymptotic Formulae for Green's Functions for the Laplacian in Domains with Small Perforations 3
1.1 Green's Function for a Multi-dimensional Domain with a Small Hole 3
1.2 Green's Function for the Dirichlet Problem in a Planar Domain with a Small Hole 9
1.2.1 Asymptotic Approximation of the Capacitary Potential 12
1.2.2 Uniform Asymptotic Approximation 13
1.3 Corollaries 17

2 Mixed and Neumann Boundary Conditions for Domains with Small Holes and Inclusions: Uniform Asymptotics of Green's Kernels 21
2.1 Mixed Boundary Value Problem in a Planar Domain with a Small Hole or a Crack 21
2.1.1 Special Solutions of Model Problems 22
2.1.2 The Dipole Matrix $\mathcal{P}$ 24
2.1.3 Pointwise Estimate of a Solution to the Exterior Neumann Problem 25
2.1.4 Asymptotic Properties of the Regular Part of the Neumann Function in $\mathbb{R}^2 \setminus F$ 27
2.1.5 Maximum Modulus Estimate for Solutions to the Mixed Problem in Ω_ε, with the Neumann Data on ∂F_ε 30
2.1.6 Approximation of Green's Function $G_\varepsilon^{(N)}$ 31
2.1.7 Simpler Asymptotic Formulae for Green's Function $G_\varepsilon^{(N)}$ 33

2.2 Mixed Boundary Value Problem with the Dirichlet Condition on ∂F_ε 35
2.2.1 Special Solutions of Model Problems 35
2.2.2 Asymptotic Property of the Regular Part of Green's Function in $\mathbb{R}^2 \setminus F$ 38
2.2.3 Maximum Modulus Estimate for Solutions to the Mixed Problem in Ω_ε, with the Dirichlet Data on ∂F_ε 39
2.2.4 Approximation of Green's Function $G_\varepsilon^{(D)}$ 40
2.2.5 Simpler Asymptotic Representation of Green's Function $G_\varepsilon^{(D)}$ 43
2.3 The Neumann Function for a Planar Domain with a Small Hole or Crack 44
2.3.1 Special Solutions of Model Problems 45
2.3.2 Maximum Modulus Estimate for Solutions to the Neumann Problem in Ω_ε 45
2.3.3 Asymptotic Approximation of N_ε 48
2.3.4 Simpler Asymptotic Representation of Neumann's Function N_ε 49
2.4 Asymptotic Approximations of Green's Kernels for Mixed and Neumann's Problems in Three Dimensions 50
2.4.1 Special Solutions of Model Problems in Limit Domains 51
2.4.2 Approximations of Green's Kernels 53

3 Green's Function for the Dirichlet Boundary Value Problem in a Domain with Several Inclusions 59
3.1 Domain of Definition and the Governing Equations for the Case of Multiple Inclusions 59
3.2 Green's Function for the Case of Anti-plane Shear for a Domain with Several Inclusions 60
3.2.1 Estimates for the Functions $h^{(j)}$ and $\zeta^{(j)}$ in the Unbounded Domain 61
3.2.2 The Capacitary Potential 61
3.2.3 A Uniform Asymptotic Approximation of Green's Function for $-\Delta$ in a Two-Dimensional Domain with Several Small Inclusions 65
3.3 Simplified Asymptotic Formulae for Green's Function Subject to Constraints on the Independent Variables 70

4 Numerical Simulations Based on the Asymptotic Approximations ... 75
4.1 Asymptotic Formulae Versus Numerical Solution for the Operator $-\Delta$ ... 75
4.1.1 Domain and the Asymptotic Approximation ... 76
4.1.2 Example: A Configuration with a Large Number of Small Inclusions ... 78
4.1.3 Example: A Configuration with Inclusions of Relatively Large Size ... 79

5 Other Examples of Asymptotic Approximations of Green's Functions in Singularly Perturbed Domains ... 83
5.1 Perturbation of a Smooth Exterior Boundary ... 83
5.2 Green's Function for the Dirichlet–Neumann Problem in a Truncated Cone ... 84
5.3 The Dirichlet–Neumann Problem in a Long Rod ... 87
5.3.1 Capacitary Potential ... 88
5.3.2 Asymptotic Approximation of Green's Function ... 89
5.3.3 Green's Function G_M Versus Green's Functions for Unbounded Domains ... 92
5.3.4 The Dirichlet–Neumann Problem in a Thin Rod ... 93

Part II Green's Tensors for Vector Elasticity in Bodies with Small Defects

6 Green's Tensor for the Dirichlet Boundary Value Problem in a Domain with a Single Inclusion ... 97
6.1 Green's Representation for Vector Elasticity ... 97
6.1.1 Geometry and Matrix Differential Operators ... 98
6.2 Estimates for the Maximum Modulus of Solutions of Elasticity Problems in Domains with Small Inclusions ... 101
6.2.1 The Maximum Principle in Ω ... 102
6.2.2 The Maximum Principle in $C\bar{\omega}$ ... 102
6.2.3 The Operator Notations ... 105
6.3 Green's Tensor for a Three-Dimensional Domain with a Small Inclusion ... 109
6.3.1 Green's Matrices for Model Domains in Three Dimensions ... 109
6.3.2 The Elastic Capacitary Potential Matrix ... 110
6.3.3 Asymptotic Estimates for the Regular Part h of Green's Tensor in an Unbounded Domain ... 118
6.3.4 A Uniform Asymptotic Formula for Green's Function G_ε in Three Dimensions ... 119
6.4 Green's Tensor for a Planar Domain with a Small Inclusion ... 124
6.4.1 Green's Kernels for Model Domains in Two Dimensions ... 124

6.4.2 Auxiliary Properties of the Regular Part h of Green's Tensor for an Unbounded Planar Domain and the Tensor ζ 125
6.4.3 A Uniform Asymptotic Approximation of an Elastic Capacitary Potential Matrix 127
6.4.4 A Uniform Asymptotic Formula for Green's Tensor G_ε in Two Dimensions 131
6.5 Simplified Asymptotic Formulae Subject to Constraints on Independent Variables for Green's Tensors in Domains with a Single Inclusion 135

7 Green's Tensor in Bodies with Multiple Rigid Inclusions 139
7.1 Estimates for Solutions of the Homogeneous Lamé Equation in a Domain with Multiple Inclusions 139
7.2 Green's Tensor for the Lamé Operator in Two-Dimensional Elasticity 144
7.2.1 Green's Matrix for a Two-Dimensional Domain with Several Small Inclusions 144
7.2.2 Green's Kernels for Model Domains in Two Dimensions 144
7.2.3 Auxiliary Matrix Functions for Two-Dimensional Elasticity 146
7.2.4 A Uniform Asymptotic Formula for Green's Tensor of Dirichlet Problem of Linear Elasticity in a Domain with Multiple Inclusions 149
7.3 Green's Matrix for a Three-Dimensional Domain with Several Small Rigid Inclusions 154
7.3.1 Green's Tensors for Model Domains in Three Dimensions 154
7.3.2 Elastic Capacitary Potential in Three Dimensions 155
7.3.3 A Uniform Asymptotic Formula for Green's Tensor in a Three-Dimensional Domain with Several Inclusions 156
7.4 Simplified Asymptotic Formulae for the Case of a Three-Dimensional Elastic Solid with Several Small Inclusions 164

8 Green's Tensor for the Mixed Boundary Value Problem in a Domain with a Small Hole 169
8.1 Definition of Green's Tensor in a Domain with a Single Void 170
8.2 An Estimate for Solutions of the Exterior Neumann Problem for the Homogeneous Lamé Equation 170
8.3 An Estimate for Solutions to the Mixed Problem for the Lamé Equation in the Perforated Domain Ω_ε 172

8.4 Model Boundary Value Problems 175
8.4.1 The Dipole Fields 176
8.4.2 The Elastic Dipole Matrix 177
8.4.3 The Asymptotics of the Matrix $\mathcal{W}$ at Infinity 178
8.4.4 The Matrix Function Υ 179
8.4.5 An Estimate for the Regular Part of the Neumann Tensor in the Unbounded Domain 179
8.5 A Uniform Asymptotic Formula for G_ε of the Mixed Problem in a Domain with a Void 183
8.6 Simplified Asymptotic Formulae for G_ε Under Constraints on the Independent Spatial Variables for a Domain with a Small Hole 186

Part III Meso-scale Approximations: Asymptotic Treatment of Perforated Domains Without Homogenization

9 Meso-scale Approximations for Solutions of Dirichlet Problems 191
9.1 Main Notations and Formulation of the Problem in the Perforated Region 191
9.2 Auxiliary Problems 193
9.2.1 Solution of the Unperturbed Problem 193
9.2.2 Capacitary Potentials of $F^{(j)}$ 193
9.2.3 Green's Function for the Unperturbed Domain 194
9.3 Formal Asymptotic Algorithm 194
9.4 Algebraic System 195
9.5 Meso-scale Uniform Approximation of u 202
9.6 The Energy Estimate 204
9.7 Meso-scale Approximation of Green's Function in Ω_N 212

10 Mixed Boundary Value Problems in Multiply-Perforated Domains 221
10.1 An Outline 222
10.2 Main Notations and Model Boundary Value Problems 223
10.3 The Formal Approximation of u_N for the Infinite Space Containing Many Voids 225
10.4 Algebraic System for the Coefficients in the Meso-scale Approximation 227
10.5 Energy Estimate 231
10.6 Approximation of u_N for a Perforated Domain 237
10.6.1 Formal Asymptotic Algorithm for the Perforated Domain Ω_N 237
10.6.2 Algebraic System 238
10.6.3 Energy Estimate for the Remainder 240
10.7 Illustrative Example 242
10.7.1 The Case of a Domain with a Cloud of Spherical Voids 242

10.7.2 Finite Elements Simulation Versus the Asymptotic Approximation 244
10.7.3 Non-uniform Cloud Containing a Large Number of Spherical Voids 245

Bibliographical Remarks 249

References 251

Subjects Index 255

Author Index 257