Symmetry Relationships between Crystal Structures

Applications of Crystallographic Group Theory in Crystal Chemistry

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with texts adapted from
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