

# **INTRODUCTION TO DIFFERENTIAL CALCULUS**

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**Systematic Studies with  
Engineering Applications  
for Beginners**

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