Table of Contents

General Introduction ........................................... xxiii

Chapter 1. Some Historical Elements ............................ 1
  1.1. Yi King .................................................. 1
  1.2. Flavor combinations in India ........................... 2
  1.3. Sand drawings in Africa ................................ 3
  1.4. Galileo’s problem ...................................... 4
  1.5. Pascal’s triangle ....................................... 7
  1.6. The combinatorial explosion: Abu Kamil’s problem, the palm grove problem and the Sudoku grid .... 9
     1.6.1. Solution to Abu Kamil’s problem ................. 11
     1.6.2. Palm Grove problem, where \( N = 4 \) ............. 12
     1.6.3. Complete Sudoku grids ............................ 14

Part 1. Combinatorics ............................................. 17

Part 1. Introduction ............................................... 19

Chapter 2. Arrangements and Combinations ...................... 21
  2.1. The three formulae ..................................... 21
  2.2. Calculation of \( C_n^p \), Pascal’s triangle and binomial formula ........................................... 25
  2.3. Exercises .............................................. 27
     2.3.1. Demonstrating formulae ........................... 27
     2.3.2. Placing rooks on a chessboard .................. 28
     2.3.3. Placing pieces on a chessboard .................. 29
     2.3.4. Pascal’s triangle modulo \( k \) ..................... 30
     2.3.5. Words classified based on their blocks of letters 31
     2.3.6. Diagonals of a polygon ........................... 33
Chapter 3. Enumerations in Alphabetical Order

3.1. Principle of enumeration of words in alphabetical order
3.2. Permutations
3.3. Writing binary numbers
  3.3.1. Programming
  3.3.2. Generalization to expression in some base $B$
3.4. Words in which each letter is less than or equal to the position
  3.4.1. Number of these words
  3.4.2. Program
3.5. Enumeration of combinations
3.6. Combinations with repetitions
3.7. Purchase of $P$ objects out of $N$ types of objects
3.8. Another enumeration of permutations
3.9. Complementary exercises
  3.9.1. Exercise 1: words with different successive letters
  3.9.2. Exercise 2: repeated purchases with a given sum of money
3.10. Return to permutations
3.11. Gray code

Chapter 4. Enumeration by Tree Structures

4.1. Words of length $n$, based on $N$ letters 1, 2, 3, ..., $N$, where each letter is followed by a higher or equal letter
4.2. Permutations enumeration
4.3. Derangements
4.4. The queens problem
4.5. Filling up containers
4.6. Stack of coins
4.7. Domino tiling a chessboard

Chapter 5. Languages, Generating Functions and Recurrences

5.1. The language of words based on two letters
5.2. Domino tiling a $2 \times n$ chessboard
5.3. Generating function associated with a sequence
5.4. Rational generating function and linear recurrence .......................... 91
5.5. Example: routes in a square grid with rising shapes without entanglement. ......................................................... 92
5.6. Exercises on recurrences .......................................................... 94
  5.6.1. Three types of purchases each day with a sum of $N$ dollars 94
  5.6.2. Word building ................................................................. 96
5.7. Examples of languages ................................................................ 98
  5.7.1. Language of parts of an element set \{a, b, c, d, \ldots\} ............. 98
  5.7.2. Language of parts of a multi-set based on $n$ elements $a$, $b$, $c$, etc., where these elements can be repeated as much as we want 99
  5.7.3. Language of words made from arrangements taken from $n$ distinct and non-repeated letters $a$, $b$, $c$, etc., where these words are shorter than or equal to $n$ ........................................................................... 99
  5.7.4. Language of words based on an alphabet of $n$ letters ............. 100
5.8. The exponential generating function ............................................. 101
  5.8.1. Exercise 1: words based on three letters $a$, $b$ and $c$, with the letter $a$ at least twice ......................................................... 101
  5.8.2. Exercise 2: sending $n$ people to three countries, with at least one person per country ......................................................... 103

Chapter 6. Routes in a Square Grid ..................................................... 105

  6.1. Shortest paths from one point to another .................................... 105
  6.2. $n$-length paths using two (perpendicular) directions of the square grid .............................................................................. 108
  6.3. Paths from $O$ to $B (n, x)$ neither touching nor crossing the horizontal axis and located above it ...................................................... 109
  6.4. Number of $n$-length paths that neither touch nor cross the axis of the adscissae until and including the final point .......................... 110
  6.5. Number of $n$-length paths above the horizontal axis that can touch but not cross the horizontal axis .............................................. 111
  6.6. Exercises .................................................................................... 112
  6.6.1. Exercise 1: show that $C_{2n}^n = \sum_{k=0}^{n} (C_n^k)^2$ ...................... 112
  6.6.2. Exercise 2: show that $\sum_{k=0}^{P} C_{N-1+k}^k = C_{N+P}$ .................. 113
  6.6.3. Exercise 3: show that $\sum_{k=0}^{n'} 2k C_{2n'}^n = n'C_{2n'}^n$ ............... 113
  6.6.4. Exercise 4: a geometrico-linguistic method ................................ 114
  6.6.5. Exercise 5: paths of a given length that never intersect each other and where the four directions are allowed in the square grid ......... 115
Chapter 7. Arrangements and Combinations with Repetitions

7.1. Anagrams
7.2. Combinations with repetitions
  7.2.1. Routes in a square grid
  7.2.2. Distributing (indiscernible) circulars in personalized letter boxes
  7.2.3. Choosing / objects out of N categories of object
  7.2.4. Number of positive or null integer solutions to the equation x0 + x1 + ... + xn-1 = P
7.3. Exercises
  7.3.1. Exercise 1: number of ways of choosing six objects out of three categories, with the corresponding prices
  7.3.2. Exercise 2: word counting
  7.3.3. Exercise 3: number of words of P characters based on an alphabet of N letters and subject to order constraints
  7.3.4. Exercise 4: choice of objects out of several categories taking at least one object from each category
  7.3.5. Exercise 5: choice of P objects out of N categories when the stock is limited
  7.3.6. Exercise 6: generating functions associated with the number of integer solutions to an equation with n unknowns
  7.3.7. Exercise 7: number of solutions to the equation x + y + z = k, where k is a given natural integer and 0 ≤ x ≤ y ≤ z
  7.3.8. Exercise 8: other applications of the method using generating functions
  7.3.9. Exercise 9: integer-sided triangles
  7.3.10. Revision exercise: sending postcards

Chapter 8. Sieve Formula

8.1. Sieve formula on sets
8.2. Sieve formula in combinatorics
8.3. Examples
  8.3.1. Example 1: filling up boxes with objects, with at least one box remaining empty
  8.3.2. Example 2: derangements
  8.3.3. Example 3: formula giving the Euler number \( \varphi(n) \)
  8.3.4. Example 4: houses to be painted
  8.3.5. Example 5: multiletter words
  8.3.6. Example 6: coloring the vertices of a graph
8.4. Exercises ................................................................. 153
  8.4.1. Exercise 1: sending nine diplomats, 1, 2, 3, ..., 9, 
to three countries A, B, C ........................................... 153
  8.4.2. Exercise 2: painting a room .................................. 153
  8.4.3. Exercise 3: rooks on a chessboard .......................... 155
8.5. Extension of sieve formula ...................................... 158
  8.5.1. Permutations that have k fixed points ..................... 159
  8.5.2. Permutations with q disjoint cycles that are k long ...... 160
  8.5.3. Terminal nodes of trees with n numbered nodes ........ 161
  8.5.4. Revision exercise about a word: intelligent ............ 163

Chapter 9. Mountain Ranges or Parenthesis Words: Catalan Numbers .... 165
  9.1. Number \( c(n) \) of mountain ranges \( 2n \) long ............ 166
  9.2. Mountains or primitive words ................................ 167
  9.3. Enumeration of mountain ranges .............................. 168
  9.4. The language of mountain ranges .............................. 169
  9.5. Generating function of the \( C_{2n} \) and Catalan numbers ... 171
  9.6. Left factors of mountain ranges .............................. 173
     9.6.1. Algorithm for obtaining the numbers of these left factors \( a(N, X) \) 175
     9.6.2. Calculation following the lines of Catalan’s triangle .. 176
     9.6.3. Calculations based on the columns of the Catalan triangle ... 177
     9.6.4. Average value of the height reached by left factors .... 178
     9.6.5. Calculations based on the second bisector of the Catalan triangle ... 180
     9.6.6. Average number of mountains for mountain ranges .... 183
  9.7. Number of peaks of mountain ranges ........................ 184
  9.8. The Catalan mountain range, its area and height ........... 187
     9.8.1. Number of mountain ranges \( 2n \) long passing through a given point 
on the square grid ............................................. 187
     9.8.2. Sum of the elements of lines in triangle \( OO'B \) of mountain 
ranges \( 2n \) long ................................................ 188
     9.8.3. Sum of numbers in triangle \( OO'B \) ........................ 189
     9.8.4. Average area of a mountain \( 2n \) long ................ 190
     9.8.5. Shape of the average mountain range .................... 192
     9.8.6. Height of the Catalan mountain range .................... 194

Chapter 10. Other Mountain Ranges ................................. 197
  10.1. Mountain ranges based on three lines \[ 
  \begin{array}{ccc}
   \hline
   \text{ } & \text{ } & \\
   \hline
   \end{array} 
  \end{matrix} \] ........................................... 197
  10.2. Words based on three lines \[ 
  \begin{array}{ccc}
   \hline
   \text{ } & \text{ } & \\
   \hline
   \end{array} 
  \end{matrix} \] with as many 
rising lines as falling lines .................................... 198
10.2.1. Explicit formula \( v(n) \) ........................................ 199
10.2.2. Return to \( u(n) \) number of mountain ranges based on three letters \( a, b, c \) and a link with \( v(n) \). ........................................ 200
10.3. Example 1: domino tiling of an enlarged Aztec diamond ........................................ 200
10.4. Example 2: domino tiling of half an Aztec diamond ........................................ 204
10.4.1. Link between Schröder numbers and Catalan numbers ........................................ 207
10.4.2. Link with Narayana numbers ........................................ 207
10.4.3. Another way of programming three-line mountain ranges ........................................ 208
10.5. Mountain ranges based on three types of lines \( \square \), \( \bigtriangleup \). ........................................ 210
10.6. Example 3: movement of the king on a chessboard ........................................ 213

Chapter 11. Some Applications of Catalan Numbers and Parenthesis Words ........................................ 215
11.1. The number of ways of placing \( n \) chords not intersecting each other on a circle with an even number \( 2n \) of points ........................................ 215
11.2. Murasaki diagrams and partitions ........................................ 216
11.3. Path couples with the same ends in a square grid ........................................ 218
11.4. Path couples with same starting point and length ........................................ 220
11.5. Decomposition of words based on two letters as a product of words linked to mountain ranges ........................................ 222

Chapter 12. Burnside's Formula ........................................ 227
12.1. Example 1: context in which we obtain the formula ........................................ 227
12.2. Burnside's formula ........................................ 231
12.2.1. Complementary exercise: rotation-type colorings of the vertices of a square ........................................ 232
12.2.2. Example 2: pawns on a chessboard ........................................ 232
12.2.3. Example 3: pearl necklaces ........................................ 237
12.2.4. Example 4: coloring of a stick ........................................ 239
12.3. Exercises ........................................ 239
12.3.1. Coloring the vertices of a square ........................................ 239
12.3.2. Necklaces with stones in several colors ........................................ 241
12.3.3. Identical balls in identical boxes ........................................ 244
12.3.4. Tiling an Aztec diamond using \( I \)-squares ........................................ 244
12.3.5. The \( 4 \times 4 \) Sudoku: search for fundamentally different symmetry-type girls ........................................ 246

Chapter 13. Matrices and Circulation on a Graph ........................................ 253
13.1. Number of paths of a given length on a complete or a regular graph ........................................ 254
13.2. Number of paths and matrix powers ........................................ 255
13.2.1. Example 1: $n$-length words in an alphabet of three letters 1, 2, 3, with prohibition of blocks 11 and 23. 257
13.2.2. Simplification of the calculation. 259
13.2.3. Example 2: $n$-length words based on three letters 1, 2, 3 with blocks 11, 22 and 33 prohibited. 261
13.3. Link between cyclic words and closed paths in an oriented graph. 262
13.4. Examples. 263
13.4.1. Dominos on a chessboard. 263
13.4.2. Words with a dependency link between two successive letters of words. 265
13.4.3. Routes on a graded segment. 266
13.4.4. Molecular chain. 270

Chapter 14. Parts and Partitions of a Set. 275
14.1. Parts of a set. 275
14.1.1. Program getting all parts of a set. 275
14.1.2. Exercises. 277
14.2. Partitions of a $n$-object set. 281
14.2.1. Definition. 281
14.2.2. A second kind of Stirling numbers, and partitions of a $n$-element set in $k$ parts. 281
14.2.3. Number of partitions of a set and Bell numbers. 283
14.2.4. Enumeration algorithm for all partitions of a set. 285
14.2.5. Exercise: Sterling numbers modulo 2. 286

Chapter 15. Partitions of a Number. 289
15.1. Enumeration algorithm. 289
15.2. Euler formula. 290
15.3. Exercises. 292
15.3.1. Exercise 1: partitions of a number $n$ in $k$ distinct elements. 292
15.3.2. Exercise 2: ordered partitions. 296
15.3.3. Exercise 3: sum of the products of all the ordered partitions of a number. 297
15.3.4. Exercise 4: partitions of a number in completely distinct parts. 298
15.3.5. Exercise 5: partitions and routes in a square grid. 299
15.3.6. Exercise 6: Ferrers graphs. 302

Chapter 16. Flags. 305
16.1. Checkered flags. 305
16.2. Flags with vertical stripes. 306
Chapter 17. Walls and Stacks ........................................ 315

17.1. Brick walls ..................................................... 315
17.2. Walls of bricks made from continuous horizontal rows .... 316
17.2.1. Algorithm for classifying various types of walls. .... 317
17.2.2. Possible positions of one row above another .......... 317
17.2.3. Coordinates of bricks ..................................... 318
17.3. Heaps ............................................................ 319
17.4. Stacks of disks ............................................... 322
17.5. Stacks of disks with continuous rows. ....................... 324
17.6. Horizontally connected polyominos .......................... 326

Chapter 18. Tiling of Rectangular Surfaces using Simple Shapes .......... 331

18.1. Tiling of a 2xn chessboard using dominos ................... 331
18.1.1. First algorithm for constructing tilings .................. 332
18.1.2. Second construction algorithm ............................ 333
18.2. Other tilings of a chessboard 2xn squares long ............. 334
18.2.1. With squares and horizontal dominos ..................... 334
18.2.2. With squares and horizontal or vertical dominos ......... 335
18.2.3. With dominos and l-squares we can turn and reflect .... 335
18.2.4. With squares, l-squares and dominos ..................... 336
18.3. Tilings of a 3xn chessboard using dominos ................. 337
18.4. Tilings of a 4xn chessboard with dominos .................. 339
18.5. Domino tilings of a rectangle ................................ 340

Chapter 19. Permutations ........................................... 345

19.1. Definition and properties .................................... 345
19.2. Decomposition of a permutation as a product of disjoint cycles ........................................... 347
19.2.1. Particular cases of permutations defined by their decomposition in cycles ............. 349
19.2.2. Number of permutations of n elements with k cycles: Stirling numbers of the first kind 352
19.2.3. Type of permutation ....................................... 353
19.3. Inversions in a permutation .................................. 354
19.3.1. Generating function of the number of inversions .... 356
19.3.2. Signature of a permutation: odd and even permutations . 357
19.4. Conjugated permutations ..................................... 359
19.5. Generation of permutations .................................. 360
19.5.1. The symmetrical group Sn is generated by the transpositions (i, j) .... 361
19.5.2. Sn is generated by transpositions of adjacent elements of the form (i, i + 1) .......... 362
19.5.3. Sn is generated by transpositions (0 1) (0 2) ... (0 n - 1) 362
19.5.4. $S_n$ is generated by cycles $(0 \, 1)$ and $(0 \, 1 \, 2 \, 3 \ldots \, n-1)$ .................................................. 363
19.6. Properties of the alternating group $A_n$ ................................................................. 363
19.6.1. $A_n$ is generated by cycles three units long: $(i \, j \, k)$ ................................. 363
19.6.2. $A_n$ is generated by $n-2$ cycles $(0 \, 1 \, k)$ .................................................. 363
19.6.3. For $n > 3$, $A_n$ is generated by the cycle chain three units long, of the form $(0 \, 1 \, 2) \, (2 \, 3 \, 4) \, (4 \, 5 \, 6) \ldots (n-3 \, n-2 \, n-1)$ ......................... 364
19.7. Applications of these properties ................................................................. 365
19.7.1. Card shuffling ........................................................................................................ 365
19.7.2. Taquin game in a $n$ by $p$ ($n$ and $p > 1$) rectangle ........................................ 368
19.7.3. Cyclic shifts in a rectangle ..................................................................................... 371
19.7.4. Exchanges of lines and columns in a square ...................................................... 375
19.8. Exercises on permutations ...................................................................................... 376
19.8.1. Creating a permutation at random ...................................................................... 376
19.8.2. Number of permutations $\begin{pmatrix} a(0) & a(1) & a(2) \ldots a(n-1) \end{pmatrix}$ with $n$ elements $0, 1, 2, \ldots, n-1$ such that $|a(i) - i| = 0$ or $1$ ..................... 377
19.8.3. Permutations with $a(i) - i = \pm 1$ or $\pm 2$ ................................................... 379
19.8.4. Permutations with $n$ elements $0, 1, 2, \ldots, n-1$ without two consecutive elements .................................................................................................................. 379
19.8.5. Permutations with $n$ elements $0, 1, 2, \ldots, n-1$, made up of a single cycle in which no two consecutive elements modulo $n$ are found ................................................................. 381
19.8.6. Involute permutations .......................................................................................... 383
19.8.7. Increasing subsequences in a permutation ........................................................ 384
19.8.8. Riffle shuffling of type $O$ and $I$ for $N$ cards when $N$ is a power of $2$ .............. 386

PART 2. PROBABILITY ........................................................................................................... 387

Part 2. Introduction ........................................................................................................... 389

Chapter 20. Reminders about Discrete Probabilities .......................................................... 395
20.1. And/or in probability theory .................................................................................... 396
20.2. Examples .................................................................................................................. 398
20.2.1. The Chevalier de Mere problem ........................................................................ 398
20.2.2. From combinatorics to probabilities ................................................................. 399
20.2.3. From combinatorics of weighted words to probabilities .................................... 400
20.2.4. Drawing a parcel of objects from a box ............................................................. 401
20.2.5. Hypergeometric law ............................................................................................ 401
20.2.6. Draws with replacement in a box ..................................................................... 402
20.2.7. Numbered balls in a box and the smallest number obtained during draws ........ 403
20.2.8. Wait for the first double heads in a repeated game of heads or tails ........................................... 404
20.2.9. Succession of random cuts made in a game of cards ................................................................. 405
20.2.10. Waiting time for initial success ........................................................................................................ 407
20.2.11. Smallest number obtained during successive draws ................................................................. 409
20.2.12. The pool problem .......................................................................................................................... 411
20.3. Total probability formula .................................................................................................................. 412
20.3.1. Classic example .................................................................................................................................. 412
20.3.2. The formula ...................................................................................................................................... 413
20.3.3. Examples ........................................................................................................................................ 413
20.4. Random variable $X$, law of $X$, expectation and variance .............................................................. 418
20.4.1. Average value of $X$ ......................................................................................................................... 418
20.4.2. Variance and standard deviation ........................................................................................................ 418
20.4.3. Example ........................................................................................................................................... 419
20.5. Some classic laws .................................................................................................................................. 420
20.5.1. Bernoulli's law .................................................................................................................................. 420
20.5.2. Geometric law .................................................................................................................................... 420
20.5.3. Binomial law ...................................................................................................................................... 421
20.6. Exercises .............................................................................................................................................. 422
20.6.1. Exercise 1: throwing balls in boxes ................................................................................................. 422
20.6.2. Exercise 2: series of repetitive tries ................................................................................................. 423
20.6.3. Exercise 3: filling two boxes ............................................................................................................ 425

Chapter 21. Chance and the Computer ........................................................................................................ 427
21.1. Random number generators ................................................................................................................. 428
21.2. Dice throwing and the law of large numbers ......................................................................................... 429
21.3. Monte Carlo methods for getting the approximate value of the number $\pi$ ...................................... 430
21.4. Average value of a random variable $X$, variance and standard deviation ......................................... 432
21.5. Computer calculation of probabilities, as well as expectation and variance, in the binomial law example ................................................................................................................................. 433
21.6. Limits of the computer ........................................................................................................................ 437
21.7. Exercises .................................................................................................................................................. 439
21.7.1. Exercise 1: throwing balls in boxes ................................................................................................. 439
21.7.2. Exercise 2: boys and girls ................................................................................................................ 439
21.7.3. Exercise 3: conditional probability ................................................................................................. 441
21.8. Appendix: chi-squared law .................................................................................................................. 443
21.8.1. Examples of the test for uniform distribution .................................................................................. 443
21.8.2. Chi-squared law and its link with Poisson distribution ................................................................... 445
Chapter 22. Discrete and Continuous ........................................... 447

22.1. Uniform law ........................................................................... 448
   22.1.1. Programming ..................................................................... 448
   22.1.2. Example 1 ....................................................................... 449
   22.1.3. Example 2: two people meeting ........................................ 450
22.2. Density function for a continuous random variable
   and distribution function ............................................................... 451
22.3. Normal law ............................................................................ 452
22.4. Exponential law and its link with uniform law ......................... 454
   22.4.1. An application: geometric law using exponential law ........ 456
   22.4.2. Program for getting the geometric law with parameter p .......... 457
22.5. Normal law as an approximation of binomial law .................... 458
22.6. Central limit theorem: from uniform law to normal law .......... 460
22.7. Appendix: the distribution function and its inversion – application
   to binomial law $B(n, p)$ ............................................................... 465
   22.7.1. Program ....................................................................... 465
   22.7.2. The inverse function ....................................................... 467
   22.7.3. Program causing us to move from distribution function
   to probability law ..................................................................... 468

Chapter 23. Generating Function Associated with a Discrete Random
Variable in a Game ........................................................................ 469

23.1. Generating function: definition and properties ....................... 469
23.2. Generating functions of some classic laws ............................... 470
   23.2.1. Bernoulli’s law ................................................................. 470
   23.2.2. Geometric law ................................................................. 470
   23.2.3. Binomial law ................................................................. 473
   23.2.4. Poisson distribution ....................................................... 475
23.3. Exercises .............................................................................. 476
   23.3.1. Exercise 1: waiting time for double heads in a game of heads
   or tails ..................................................................................... 476
   23.3.2. Exercise 2: in a repeated game of heads or tails, what is the parity
   of the number of heads? ......................................................... 481
   23.3.3. Exercise 3: draws until a certain threshold is exceeded ........ 482
   23.3.4. Exercise 4: Pascal’s law ................................................... 487
   23.3.5. Exercise 5: balls of two colors in a box ............................ 488
   23.3.6. Exercise 6: throws of $N$ dice until each gives the number 1 ... 492


24.1. First example: counting of words based on three letters .......... 497
24.2. Generating functions and determinants ................................ 499
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>24.3</td>
<td>Examples</td>
<td>500</td>
</tr>
<tr>
<td>24.3.1</td>
<td>Exercise 1: waiting time for double heads in a game of heads or tails</td>
<td>500</td>
</tr>
<tr>
<td>24.3.2</td>
<td>Draws from three boxes</td>
<td>503</td>
</tr>
<tr>
<td>24.3.3</td>
<td>Alternate draws from two boxes</td>
<td>505</td>
</tr>
<tr>
<td>24.3.4</td>
<td>Successive draws from one box to the next</td>
<td>506</td>
</tr>
<tr>
<td>25.1</td>
<td>Paths on a square grid</td>
<td>509</td>
</tr>
<tr>
<td>25.2</td>
<td>Probability of getting a certain number of wins after ( n ) equiprobable tosses</td>
<td>511</td>
</tr>
<tr>
<td>25.2.1</td>
<td>Probability ( p(n, x) ) of getting winnings of ( x ) at the end of ( n ) moves</td>
<td>512</td>
</tr>
<tr>
<td>25.2.2</td>
<td>Standard deviation in relation to a starting point</td>
<td>512</td>
</tr>
<tr>
<td>25.2.3</td>
<td>Probability ( \alpha(2n') ) of a return to the origin at stage ( n = 2n' )</td>
<td>513</td>
</tr>
<tr>
<td>25.3</td>
<td>Probabilities of certain routes over ( n ) moves</td>
<td>514</td>
</tr>
<tr>
<td>25.4</td>
<td>Complementary exercises</td>
<td>516</td>
</tr>
<tr>
<td>25.4.1</td>
<td>Last visit to the origin</td>
<td>516</td>
</tr>
<tr>
<td>25.4.2</td>
<td>Number of winnings sign changes throughout the game</td>
<td>517</td>
</tr>
<tr>
<td>25.4.3</td>
<td>Probability of staying on the positive winnings side for a certain amount of time during the ( N = 2n' ) equiprobable tosses</td>
<td>519</td>
</tr>
<tr>
<td>25.4.4</td>
<td>Longest range of winnings with constant sign</td>
<td>520</td>
</tr>
<tr>
<td>25.5</td>
<td>The gambler's ruin problem</td>
<td>521</td>
</tr>
<tr>
<td>25.5.1</td>
<td>Probability of ruin</td>
<td>522</td>
</tr>
<tr>
<td>25.5.2</td>
<td>Average duration of the game</td>
<td>524</td>
</tr>
<tr>
<td>25.5.3</td>
<td>Results and program</td>
<td>525</td>
</tr>
<tr>
<td>25.5.4</td>
<td>Exercises</td>
<td>526</td>
</tr>
<tr>
<td>25.5.5</td>
<td>Temperature equilibrium and random walk</td>
<td>530</td>
</tr>
<tr>
<td>26.1</td>
<td>Movement of a particle on a polygon or graduated segment</td>
<td>535</td>
</tr>
<tr>
<td>26.1.1</td>
<td>Average duration of routes between two points</td>
<td>535</td>
</tr>
<tr>
<td>26.1.2</td>
<td>Paths of a given length on a polygon</td>
<td>542</td>
</tr>
<tr>
<td>26.1.3</td>
<td>Particle circulating on a pentagon: time required using one side or the other to get to the end</td>
<td>546</td>
</tr>
<tr>
<td>26.2</td>
<td>Movement on a polyhedron</td>
<td>547</td>
</tr>
<tr>
<td>26.2.1</td>
<td>Case of the regular polyhedron</td>
<td>547</td>
</tr>
<tr>
<td>26.2.2</td>
<td>Circulation on a cube with any dimensions</td>
<td>550</td>
</tr>
<tr>
<td>26.3</td>
<td>The robot and the human being</td>
<td>555</td>
</tr>
<tr>
<td>26.4</td>
<td>Exercises</td>
<td>559</td>
</tr>
<tr>
<td>26.4.1</td>
<td>Movement of a particle on a square-based pyramid</td>
<td>559</td>
</tr>
<tr>
<td>26.4.2</td>
<td>Movement of two particles on a square-based pyramid</td>
<td>561</td>
</tr>
<tr>
<td>26.4.3</td>
<td>Movement of two particles on a graph with five vertices</td>
<td>563</td>
</tr>
</tbody>
</table>
Chapter 27. Repetitive Draws until the Outcome of a Certain Pattern . . . 565

27.1. Patterns are arrangements of K out of N letters .......................... 566
27.1.1. Wait for a given arrangement of the K letters in the form of a block .............................................. 566
27.1.2. Wait for a given cyclic arrangement of K letters in the form of a block .............................................. 568
27.1.3. The pattern is a given arrangement of K out of N letters in scattered form .............................................. 570
27.2. Patterns are combinations of K letters drawn from N letters ......... 571
27.2.1. Wait for the outcome of a part made of K numbers in the form of a block .............................................. 571
27.2.2. Wait for the outcome of any part of K numbers in the form of a block, out of N .............................................. 574
27.2.3. Wait for the outcome of a part with K given numbers out of N in scattered form .............................................. 577
27.2.4. Wait for the outcome of any part of K numbers out of N, in scattered form .............................................. 577
27.2.5. Some examples of comparative results for waiting times ......... 579
27.3. Wait for patterns with eventual repetitions of identical letters ....... 580
27.3.1. For an alphabet of N letters, we wait for a given pattern in the form of a w-length block .............................................. 580
27.3.2. Wait for one of two patterns of the same length L .................. 581
27.4. Programming exercises ......................................................... 586
27.4.1. Wait for completely different letters ................................. 586
27.4.2. Waiting time for a certain pattern ..................................... 588
27.4.3. Number of words without two-sided factors ....................... 589

Chapter 28. Probability Exercises ................................................. 597

28.1. The elevator ................................................................. 597
28.1.1. Deal with the case where P = 2 floors and the number of people N is at least equal to 2 .......................... 597
28.1.2. Determine the law of X, i.e. the probability associated with each value of X .............................................. 598
28.1.3. Average value E(X) ..................................................... 599
28.1.4. Direct calculation of S(K+1, K) ..................................... 600
28.1.5. Another way of dealing with the previous question .......... 601
28.2. Matches ................................................................. 601
28.3. The tunnel ................................................................. 602
28.3.1. Dealing with the specific case where N = 3 ......................... 606
28.3.2. Variation with an absorbing boundary and another method .... 608
28.3.3. Complementary exercise: drunken man's walk on a straight line, with resting time .......................... 610
28.4. Repetitive draws from a box ........................................... 613
28.4.1. Probability law for the number of draws ................. 615
28.4.2. Extra questions ..................................................... 616
28.4.3. Probability of getting ball number $k$ during the game .... 617
28.4.4. Probability law associated with the number of balls drawn ... 617
28.4.5. Complementary exercise: variation of the previous problem ... 618
28.5. The sect ................................................................. 620
28.5.1. Can the group last forever? .................................. 620
28.5.2. Probability law of the size of the tree ..................... 621
28.5.3. Average tree size ................................................. 622
28.5.4. Variance of the variable size ................................. 624
28.5.5. Algorithm giving the probability law of the organization's lifespan ................................................. 625
28.6. Surfing the web (or how Google works) ......................... 627

PART 3. GRAPHS ................................................................. 637

Part 3. Introduction ......................................................... 639

Chapter 29. Graphs and Routes ........................................... 643
29.1. First notions on graphs ............................................. 643
29.1.1. A few properties of graphs .................................. 645
29.1.2. Constructing graphs from points ......................... 646
29.2. Representing a graph in a program ............................ 647
29.2.1. From vertices to edges ...................................... 649
29.2.2. From edges to vertices ...................................... 649
29.3. The tree as a specific graph .................................... 649
29.3.1. Definitions and properties ................................ 649
29.3.2. Programming exercise: network converging on a point .... 652
29.4. Paths from one point to another in a graph .................. 654
29.4.1. Dealing with an example ................................. 654
29.4.2. Exercise: paths on a complete graph, from one vertex to another .......................... 656

Chapter 30. Explorations in Graphs ....................................... 661
30.1. The two ways of visiting all the vertices of a connected graph .......................... 661
30.2. Visit to all graph nodes from one node, following depth-first traversal .................. 662
30.3. The pedestrian's route ........................................... 665
30.4. Depth-first exploration to determine connected components of the graph .................. 669
30.5. Breadth-first traversal ........................................... 671
30.5.1. Program ......................................................... 671
30.5.2. Example: traversal in a square grid. ................................................. 673
30.6. Exercises ............................................................................................... 676
30.6.1. Searching in a maze. ........................................................................ 676
30.6.2. Routes in a square grid, with rising shapes without entangling ......... 680
30.6.3. Route of a fluid in a graph ................................................................. 683
30.6.4. Connected graphs with \( n \) vertices ................................................. 683
30.6.5. Bipartite graphs ................................................................................. 685
30.7. Returning to a depth-first exploration tree .......................................... 686
30.7.1. Returning edges in an undirected graph ............................................ 687
30.7.2. Isthmus in an undirected graph ......................................................... 688
30.8. Case of directed graphs ......................................................................... 690
30.8.1. Strongly connected components in a directed graph ......................... 690
30.8.2. Transitive closure of a directed graph .............................................. 693
30.8.3. Orientation of a connected undirected graph to become strongly connected ................................................................. 696
30.8.4. The best orientations on a graph ....................................................... 696
30.9. Appendix: constructing the maze (simplified version) ......................... 700

Chapter 31. Trees with Numbered Nodes, Cayley's Theorem and Prüfer Code ................................................................. 705
31.1. Cayley's theorem .................................................................................. 705
31.2. Prüfer code .......................................................................................... 706
31.2.1. Passage from a tree to its Prüfer code .............................................. 707
31.2.2. Reverse process ............................................................................... 707
31.2.3. Program ......................................................................................... 709
31.3. Randomly constructed spanning tree ................................................... 715
31.3.1. Wilson's algorithm ......................................................................... 715
31.3.2. Maze and domino tiling .................................................................. 718

Chapter 32. Binary Trees ................................................................................ 723
32.1. Number of binary trees with \( n \) nodes ..................................................... 725
32.2. The language of binary trees ................................................................. 725
32.3. Algorithm for creation of words from the binary tree language .......... 728
32.4. Triangulation of polygons with numbered vertices and binary trees. .... 729
32.5. Binary tree sort or quicksort ................................................................... 733

Chapter 33. Weighted Graphs: Shortest Paths and Minimum Spanning Tree ............................................................................ 737
33.1. Shortest paths in a graph ....................................................................... 737
33.1.1. Dijkstra's algorithm ......................................................................... 738
33.1.2. Floyd's algorithm ............................................................................. 741
33.2. Minimum spanning tree ....................................................................... 746
33.2.1. Prim's algorithm .............................................. 747
33.2.2. Kruskal's algorithm ........................................... 749
33.2.3. Comparison of the two algorithms ......................... 754
33.2.4. Exercises ...................................................... 754

Chapter 34. Eulerian Paths and Cycles, Spanning Trees of a Graph .... 759
34.1. Definition of Eulerian cycles and paths ....................... 759
34.2. Euler and Königsberg bridges .................................. 761
  34.2.1. Returning to Königsberg bridges ......................... 763
  34.2.2. Examples .................................................... 764
  34.2.3. Constructing Eulerian cycles by fusing cycles .......... 767
34.3. Number of Eulerian cycles in a directed graph, link with directed
  spanning trees ..................................................... 768
  34.3.1. Number of directed spanning trees ....................... 771
  34.3.2. Examples ................................................... 774
34.4. Spanning trees of an undirected graph ........................ 776
  34.4.1. Example 1: complete graph with p vertices ............. 777
  34.4.2. Example 2: tetrahedron .................................. 778

Chapter 35. Enumeration of Spanning Trees of an Undirected Graph .... 779
35.1. Spanning trees of the fan graph .............................. 779
35.2. The ladder graph and its spanning trees ...................... 782
35.3. Spanning trees in a square network in the form of a grid .... 784
  35.3.1. Experimental enumeration of spanning trees
          of the square network ....................................... 785
  35.3.2. Spanning trees program in the case of the square network
          ............................................................. 786
  35.3.3. Passage to the undirected graph, its dual and formula giving the
          number of spanning trees .................................. 788
35.4. The two essential types of (undirected) graphs based on squares .. 789
35.5. The cyclic square graph ....................................... 791
35.6. Examples of regular graphs ................................... 792
  35.6.1. Example 1 .................................................. 792
  35.6.2. Example 2: hypercube with n dimensions ................ 793
  35.6.3. Example 3: the ladder graph and its variations ........ 793

Chapter 36. Enumeration of Eulerian Paths in Undirected Graphs ....... 799
36.1. Polygon graph with n vertices with double edges ............. 799
36.2. Eulerian paths in graph made up of a frieze of triangles ....... 801
36.3. Algorithm for Eulerian paths and cycles on an undirected graph 804
  36.3.1. The arborescence for the paths ........................ 804
  36.3.2. Program for enumerating Eulerian cycles ............... 805
### Table of Contents

36.3.3. Enumeration in the case of multiple edges between vertices. ........................................... 807
36.3.4. Another example: square with double diagonals .......................................................... 810
36.4. The game of dominos ........................................................................................................... 813
36.4.1. Number of domino chains .............................................................................................. 813
36.4.2. Algorithms ....................................................................................................................... 816
36.5. Congo graphs ...................................................................................................................... 820
36.5.1. A simple case: graphs $P(2n, 5)$ .................................................................................. 822
36.5.2. The first type of Congolese drawings, on $P(n + 1, n)$ graphs, with their Eulerian paths .. 826
36.5.3. The second type of Congolese drawings, on $P(2N, N)$ graphs .................................... 826
36.5.4. Case of Eulerian cycles on $P(2N + 1, 2N - 1)$ graphs .............................................. 830
36.5.5. Case of $I(2N + 1, 2N + 1)$ graphs with their Eulerian cycles ................................. 832

**Chapter 37. Hamiltonian Paths and Circuits** ........................................................................... 835

37.1. Presence or absence of Hamiltonian circuits ................................................................. 836
37.1.1. First examples ................................................................................................................. 836
37.1.2. Hamiltonian circuits on a cube ...................................................................................... 837
37.1.3. Complete graph and Hamiltonian circuits ................................................................. 839
37.2. Hamiltonian circuits covering a complete graph ............................................................. 840
37.2.1. Case where the number of vertices is a prime number other than two ....................... 840
37.2.2. General case ..................................................................................................................... 841
37.3. Complete and antisymmetric directed graph ................................................................. 843
37.3.1. A few theoretical considerations .................................................................................... 843
37.3.2. Experimental verification and algorithms ...................................................................... 848
37.3.3. Complete treatment of case $N = 4$ ............................................................................. 851
37.4. Bipartite graph and Hamiltonian paths ............................................................................ 854
37.5. Knights tour graph on the $N \times N$ chessboard ............................................................ 855
37.5.1. Case where $N$ is odd ..................................................................................................... 855
37.5.2. Coordinates of the neighbors of a vertex ...................................................................... 855
37.5.3. Hamiltonian cycles program ......................................................................................... 856
37.5.4. Another algorithm ......................................................................................................... 857
37.6. de Bruijn sequences ........................................................................................................... 859
37.6.1. Preparatory example ....................................................................................................... 859
37.6.2. Definition ........................................................................................................................ 860
37.6.3. de Bruijn graph ............................................................................................................... 862
37.6.4. Number of Eulerian and Hamiltonian cycles of $G_n$ .................................................. 865

**APPENDICES** .......................................................................................................................... 867

**Appendix 1. Matrices** .............................................................................................................. 869

A1.1. Notion of linear application .............................................................................................. 869
A1.2. Bijective linear application .......................................................... 872
A1.3. Base change .................................................................................. 873
A1.4. Product of two matrices ................................................................ 874
A1.5. Inverse matrix .............................................................................. 875
A1.6. Eigenvalues and eigenvectors ......................................................... 877
A1.7. Similar matrices ............................................................................ 879
A1.8. Exercise ......................................................................................... 881
A1.9. Eigenvalues of circulant matrices and circular graphs .................. 882

Appendix 2. Determinants and Route Combinatorics ......................... 885
A2.1. Recalling determinants ................................................................. 885
A2.2. Determinants and tilings ............................................................... 887
A2.3. Path sets and determinant ............................................................. 892
   A2.3.1. First example: paths without intersection in a square network . 892
   A2.3.2. Second example: mountain ranges without intersection,
          based on two diagonal lines ......................................................... 895
   A2.3.3. Third example: mountain ranges without intersection based on
          diagonal lines and plateaus. Link with Aztec diamond tilings ....... 896
A2.3.4. Diamond tilings ....................................................................... 899
A2.4. The hamburger graph: disjoint cycles ......................................... 901
   A2.4.1. First example: domino tiling of a rectangular checkerboard
          \(N\) long, 2 wide ................................................................. 902
   A2.4.2. Second example: domino tilings of the Aztec diamond ......... 904

Bibliography ....................................................................................... 907

Index .................................................................................................. 911