ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Boolean Functions

Theory, Algorithms, and Applications

YVES CRAMA
University of Liège, Belgium

PETER L. HAMMER

Contents

Contributors *page* xiii
Preface xv
Acknowledgments xix
Notations xxi

Part I Foundations

1 Fundamental concepts and applications 3
1.1 Boolean functions: Definitions and examples 3
1.2 Boolean expressions 8
1.3 Duality 13
1.4 Normal forms 14
1.5 Transforming an arbitrary expression into a DNF 19
1.6 Orthogonal DNFs and number of true points 22
1.7 Implicants and prime implicants 24
1.8 Restrictions of functions, essential variables 28
1.9 Geometric interpretation 31
1.10 Monotone Boolean functions 33
1.11 Recognition of functional and DNF properties 40
1.12 Other representations of Boolean functions 44
1.13 Applications 49
1.14 Exercises 65

2 Boolean equations 67
2.1 Definitions and applications 67
2.2 The complexity of Boolean equations: Cook's theorem 72
2.3 On the role of DNF equations 74
2.4 What does it mean to "solve a Boolean equation"? 78
2.5 Branching procedures 80
2.6 Variable elimination procedures 87

2.7 The consensus procedure 92
2.8 Mathematical programming approaches 95
2.9 Recent trends and algorithmic performance 103
2.10 More on the complexity of Boolean equations 104
2.11 Generalizations of consistency testing 111
2.12 Exercises 121

3 Prime implicants and minimal DNFs 123
Peter L. Hammer and Alexander Kogan
3.1 Prime implicants 123
3.2 Generation of all prime implicants 128
3.3 Logic minimization 141
3.4 Extremal and typical parameter values 159
3.5 Exercises 165

4 Duality theory 167
Yves Crama and Kazuhisa Makino
4.1 Basic properties and applications 167
4.2 Duality properties of positive functions 176
4.3 Algorithmic aspects: The general case 183
4.4 Algorithmic aspects: Positive functions 189
4.5 Exercises 198

Part II Special Classes

5 Quadratic functions 203
Bruno Simeone
5.1 Basic definitions and properties 203
5.2 Why are quadratic Boolean functions important? 205
5.3 Special classes of quadratic functions 207
5.4 Quadratic Boolean functions and graphs 209
5.5 Reducibility of combinatorial problems to quadratic equations 218
5.6 Efficient graph-theoretic algorithms for quadratic equations 230
5.7 Quadratic equations: Special topics 243
5.8 Prime implicants and irredundant forms 250
5.9 Dualization of quadratic functions (Contributed by Oya Ekin Karaşan) 263
5.10 Exercises 266

6 Horn functions 269
Endre Boros
6.1 Basic definitions and properties 269
6.2 Applications of Horn functions 273
6.3 False points of Horn functions 277

6.4 Horn equations 281
6.5 Prime implicants of Horn functions 286
6.6 Properties of the set of prime implicants 292
6.7 Minimization of Horn DNFs 297
6.8 Dualization of Horn functions 306
6.9 Special classes 309
6.10 Generalizations 314
6.11 Exercises 321

7 Orthogonal forms and shellability 326
7.1 Computation of orthogonal DNFs 326
7.2 Shellings and shellability 330
7.3 Dualization of shellable DNFs 336
7.4 The lexico-exchange property 338
7.5 Shellable quadratic DNFs and graphs 346
7.6 Applications 348
7.7 Exercises 349

8 Regular functions 351
8.1 Relative strength of variables and regularity 351
8.2 Basic properties 355
8.3 Regularity and left-shifts 362
8.4 Recognition of regular functions 365
8.5 Dualization of regular functions 369
8.6 Regular set covering problems 377
8.7 Regular minorants and majorants 380
8.8 Higher-order monotonicity 391
8.9 Generalizations of regularity 397
8.10 Exercises 401

9 Threshold functions 404
9.1 Definitions and applications 404
9.2 Basic properties of threshold functions 408
9.3 Characterizations of threshold functions 413
9.4 Recognition of threshold functions 417
9.5 Prime implicants of threshold functions 423
9.6 Chow parameters of threshold functions 428
9.7 Threshold graphs 438
9.8 Exercises 444

10 Read-once functions 448
Martin C. Golumbic and Vladimir Gurvich
10.1 Introduction 448
10.2 Dual implicants 450

10.3 Characterizing read-once functions 456
10.4 The properties of P_4-free graphs and cographs 463
10.5 Recognizing read-once functions 466
10.6 Learning read-once functions 473
10.7 Related topics and applications of read-once functions 476
10.8 Historical notes 480
10.9 Exercises 481

11 Characterizations of special classes by functional equations 487
Lisa Hellerstein
11.1 Characterizations of positive functions 487
11.2 Functional equations 488
11.3 Characterizations of particular classes 491
11.4 Conditions for characterization 495
11.5 Finite characterizations by functional equations 500
11.6 Exercises 506

Part III Generalizations

12 Partially defined Boolean functions 511
Toshihide Ibaraki
12.1 Introduction 511
12.2 Extensions of pdBfs and their representations 514
12.3 Extensions within given function classes 531
12.4 Best-fit extensions of pdBfs containing errors 547
12.5 Extensions of pdBfs with missing bits 551
12.6 Minimization with don't cares 558
12.7 Conclusion 561
12.8 Exercises 562

13 Pseudo-Boolean functions 564
13.1 Definitions and examples 564
13.2 Representations 570
13.3 Extensions of pseudo-Boolean functions 578
13.4 Pseudo-Boolean optimization 585
13.5 Approximations 593
13.6 Special classes of pseudo-Boolean functions 593
13.7 Exercises 607

A Graphs and hypergraphs 609
A.1 Undirected graphs 609
A.2 Directed graphs 612
A.3 Hypergraphs 614

B Algorithmic complexity 615
B.1 Decision problems 615
B.2 Algorithms 617
B.3 Running time, polynomial-time algorithms, and the class P 618
B.4 The class NP 619
B.5 Polynomial-time reductions and NP-completeness 620
B.6 The class co-NP 621
B.7 Cook's theorem 622
B.8 Complexity of list-generation and counting algorithms 624

C JBool: A software tool 627
Claude Benzaken and Nadia Brauner
C.1 Introduction 627
C.2 Work interface 628
C.3 Creating a Boolean function 629
C.4 Editing a function 632
C.5 Operations on Boolean functions 633

Bibliography 635
Index 677