

Discrete Harmonic Analysis

Representations, Number Theory, Expanders, and the Fourier Transform

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