Critical Phenomena in Natural Sciences

Chaos, Fractals, Selforganization and Disorder: Concepts and Tools

Second Edition
With 102 Figures
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