CONTENTS

Preface to A K Peters Edition xv
Preface xvii
1 Introduction to Asymptotic Analysis

1 Origin of Asymptotic Expansions 1
2 The Symbols ~, o, and O 4
3 The Symbols ~, o, and O (continued) 6
4 Integration and Differentiation of Asymptotic and Order Relations 8
5 Asymptotic Solution of Transcendental Equations: Real Variables 11
6 Asymptotic Solution of Transcendental Equations: Complex Variables 14
7 Definition and Fundamental Properties of Asymptotic Expansions 16
8 Operations with Asymptotic Expansions 19
9 Functions Having Prescribed Asymptotic Expansions 22
10 Generalizations of Poincaré's Definition 24
11 Error Analysis; Variational Operator Historical Notes and Additional References 27

2 Introduction to Special Functions

1 The Gamma Function 31
2 The Psi Function 39
3 Exponential, Logarithmic, Sine, and Cosine Integrals 40
4 Error Functions, Dawson's Integral, and Fresnel Integrals 43
5 Incomplete Gamma Functions 45
6 Orthogonal Polynomials 46
7 The Classical Orthogonal Polynomials 48
8 The Airy Integral 53
## 3 Integrals of a Real Variable

1. Integration by Parts 66
2. Laplace Integrals 67
3. Watson's Lemma 71
4. The Riemann–Lebesgue Lemma 73
5. Fourier Integrals 75
6. Examples; Cases of Failure 76
7. Laplace's Method 80
8. Asymptotic Expansions by Laplace's Method; Gamma Function of Large Argument 85
9. Error Bounds for Watson's Lemma and Laplace's Method 89
10. Examples 92
11. The Method of Stationary Phase 96
12. Preliminary Lemmas 98
13. Asymptotic Nature of the Stationary Phase Approximation 100
14. Asymptotic Expansions by the Method of Stationary Phase 104

## 4 Contour Integrals

1. Laplace Integrals with a Complex Parameter 106
2. Incomplete Gamma Functions of Complex Argument 109
3. Watson's Lemma 112
4. Airy Integral of Complex Argument; Compound Asymptotic Expansions 116
5. Ratio of Two Gamma Functions; Watson's Lemma for Loop Integrals 118
7. Saddle Points 125
8. Examples 127
9. Bessel Functions of Large Argument and Order 130
10. Error Bounds for Laplace's Method; the Method of Steepest Descents 135

Historical Notes and Additional References 137
5 Differential Equations with Regular Singularities; Hypergeometric and Legendre Functions

1 Existence Theorems for Linear Differential Equations: Real Variables 139
2 Equations Containing a Real or Complex Parameter 143
3 Existence Theorems for Linear Differential Equations: Complex Variables 145
4 Classification of Singularities; Nature of the Solutions in the Neighborhood of a Regular Singularity 148
5 Second Solution When the Exponents Differ by an Integer or Zero 150
6 Large Values of the Independent Variable 153
7 Numerically Satisfactory Solutions 154
8 The Hypergeometric Equation 156
9 The Hypergeometric Function 159
10 Other Solutions of the Hypergeometric Equation 163
11 Generalized Hypergeometric Functions 168
12 The Associated Legendre Equation 169
13 Legendre Functions of General Degree and Order 174
14 Legendre Functions of Integer Degree and Order 180
15 Ferrers Functions 185

Historical Notes and Additional References 189

6 The Liouville–Green Approximation

1 The Liouville Transformation 190
2 Error Bounds: Real Variables 193
3 Asymptotic Properties with Respect to the Independent Variable 197
4 Convergence of \( \psi (F) \) at a Singularity 200
5 Asymptotic Properties with Respect to Parameters 203
6 Example: Parabolic Cylinder Functions of Large Order 206
7 A Special Extension 208
8 Zeros 211
9 Eigenvalue Problems 214
10 Theorems on Singular Integral Equations 217
11 Error Bounds: Complex Variables 220
12 Asymptotic Properties for Complex Variables 223
13 Choice of Progressive Paths 224

Historical Notes and Additional References 228
7 Differential Equations with Irregular Singularities; Bessel and Confluent Hypergeometric Functions

1 Formal Series Solutions 229
2 Asymptotic Nature of the Formal Series 232
3 Equations Containing a Parameter 236
4 Hankel Functions; Stokes' Phenomenon 237
5 The Function \( Y_v(z) \) 241
6 Zeros of \( J_v(z) \) 244
7 Zeros of \( Y_v(z) \) and Other Cylinder Functions 248
8 Modified Bessel Functions 250
9 Confluent Hypergeometric Equation 254
10 Asymptotic Solutions of the Confluent Hypergeometric Equation 256
11 Whittaker Functions 260
12 Error Bounds for the Asymptotic Solutions in the General Case 262
13 Error Bounds for Hankel's Expansions 266
14 Inhomogeneous Equations 270
15 Struve's Equation 274
    Historical Notes and Additional References 277

8 Sums and Sequences

1 The Euler-Maclaurin Formula and Bernoulli's Polynomials 279
2 Applications 284
3 Contour Integral for the Remainder Term 289
4 Stirling's Series for \( \ln \Gamma(z) \) 293
5 Summation by Parts 295
6 Barnes' Integral for the Hypergeometric Function 299
7 Further Examples 302
8 Asymptotic Expansions of Entire Functions 307
9 Coefficients in a Power-Series Expansion; Method of Darboux 309
10 Examples 311
11 Inverse Laplace Transforms; Haar's Method 315
    Historical Notes and Additional References 321

9 Integrals: Further Methods

1 Logarithmic Singularities 322
2 Generalizations of Laplace's Method 325
3 Example from Combinatoric Theory 329
4 Generalizations of Laplace's Method (continued) 331
### Differential Equations with a Parameter: Expansions in Elementary Functions

1. Classification and Preliminary Transformations 362
2. Case I: Formal Series Solutions 364
3. Error Bounds for the Formal Solutions 366
4. Behavior of the Coefficients at a Singularity 368
5. Behavior of the Coefficients at a Singularity (continued) 369
6. Asymptotic Properties with Respect to the Parameter 371
7. Modified Bessel Functions of Large Order 374
8. Extensions of the Regions of Validity for the Expansions of the Modified Bessel Functions 378
9. More General Forms of Differential Equation 382
10. Inhomogeneous Equations 386

#### Example: An Inhomogeneous Form of the Modified Bessel Equation 388

### Historical Notes and Additional References 391

### Differential Equations with a Parameter: Turning Points

1. Airy Functions of Real Argument 392
2. Auxiliary Functions for Real Variables 394
3. The First Approximation 397
4. Asymptotic Properties of the Approximation; Whittaker Functions with $m$ Large 401
5. Real Zeros of the Airy Functions 403
6. Zeros of the First Approximation 405
7. Higher Approximations 408
8. Airy Functions of Complex Argument 413
9. Asymptotic Approximations for Complex Variables 416
10. Bessel Functions of Large Order 419

### Historical Notes and Additional References 419
## 11 More General Form of Differential Equation

426

## 12 Inhomogeneous Equations

429

Historical Notes and Additional References

433

### 12 Differential Equations with a Parameter: Simple Poles and Other Transition Points

1. **Bessel Functions and Modified Bessel Functions of Real Order and Argument**
   - 435
2. **Case III: Formal Series Solutions**
   - 438
3. **Error Bounds: Positive $\zeta$**
   - 440
4. **Error Bounds: Negative $\zeta$**
   - 443
5. **Asymptotic Properties of the Expansions**
   - 447
6. **Determination of Phase Shift**
   - 449
7. **Zeros**
   - 451
8. **Auxiliary Functions for Complex Arguments**
   - 453
9. **Error Bounds: Complex $u$ and $\zeta$**
   - 457
10. **Asymptotic Properties for Complex Variables**
    - 460
11. **Behavior of the Coefficients at Infinity**
    - 462
12. **Legendre Functions of Large Degree: Real Arguments**
    - 463
13. **Legendre Functions of Large Degree: Complex Arguments**
    - 470
14. **Other Types of Transition Points**
    - 474
Historical Notes and Additional References

478

### 13 Connection Formulas for Solutions of Differential Equations

1. **Introduction**
   - 480
2. **Connection Formulas at a Singularity**
   - 480
3. **Differential Equations with a Parameter**
   - 482
4. **Connection Formula for Case III**
   - 483
5. **Application to Simple Poles**
   - 487
6. **Example: The Associated Legendre Equation**
   - 490
7. **The Gans-Jeffreys Formulas: Real-Variable Method**
   - 491
8. **Two Turning Points**
   - 494
9. **Bound States**
   - 497
10. **Wave Penetration through a Barrier. I**
    - 501
11. **Fundamental Connection Formula for a Simple Turning Point in the Complex Plane**
    - 503
12. **Example: Airy's Equation**
    - 507
13. **Choice of Progressive Paths**
    - 508
    - 510
15. **Wave Penetration through a Barrier. II**
    - 513
Historical Notes and Additional References

516
## 14 Estimation of Remainder Terms

1 Numerical Use of Asymptotic Approximations 519
2 Converging Factors 522
3 Exponential Integral 523
4 Exponential Integral (continued) 527
5 Confluent Hypergeometric Function 531
6 Euler's Transformation 536
7 Application to Asymptotic Expansions 540
   Historical Notes and Additional References 543

### Answers to Exercises 545

### References 548

### Index of Symbols 561

### General Index 563