Numerical Methods for Nonlinear Elliptic Differential Equations

A Synopsis

Klaus Böhmer
University of Marburg
Contents

Preface xiv

PART I  ANALYTICAL RESULTS

1 From linear to nonlinear equations, fundamental results 3
  1.1 Introduction 3
  1.2 Linear versus nonlinear models 3
  1.3 Examples for nonlinear partial differential equations 10
  1.4 Fundamental results 13
    1.4.1 Linear operators and functionals in Banach spaces 13
    1.4.2 Inequalities and $L^p(\Omega)$ spaces 18
    1.4.3 Hölder and Sobolev spaces and more 20
    1.4.4 Derivatives in Banach spaces 27

2 Elements of analysis for linear and nonlinear partial elliptic
differential equations and systems 32
  2.1 Introduction 32
  2.2 Linear elliptic differential operators of second order, bilinear forms
    and solution concepts 36
  2.3 Bilinear forms and induced linear operators 45
  2.4 Linear elliptic differential operators, Fredholm alternative and regular
    solutions 54
    2.4.1 Introduction 54
    2.4.2 Linear operators of order $2m$ with $C^\infty$ coefficients 58
    2.4.3 Linear operators of order 2 under $C^k$ conditions 64
    2.4.4 Weak elliptic equation of order $2m$ in Hilbert spaces 69
  2.5 Nonlinear elliptic equations 77
    2.5.1 Introduction 77
    2.5.2 Definitions for nonlinear elliptic operators 79
    2.5.3 Special semilinear and quasilinear operators 81
    2.5.4 Quasilinear elliptic equations of order 2 88
    2.5.5 General nonlinear and Nemyckii operators 96
    2.5.6 Divergent quasilinear elliptic equations of order $2m$ 100
    2.5.7 Fully nonlinear elliptic equations of orders 2, $m$ and $2m$ 108
  2.6 Linear and nonlinear elliptic systems 113
    2.6.1 Introduction 113
    2.6.2 General systems of elliptic differential equations 114
    2.6.3 Linear elliptic systems of order 2 118
2.6.4 Quasilinear elliptic systems of order 2 and variational methods 125
2.6.5 Linear elliptic systems of order $2m, m \geq 1$ 132
2.6.6 Divergent quasilinear elliptic systems of order $2m$ 137
2.6.7 Nemyckii operators and quasilinear divergent systems of order $2m$ 140
2.6.8 Fully nonlinear elliptic systems of orders 2 and $2m$ 146

2.7 Linearization of nonlinear operators 147
2.7.1 Introduction 147
2.7.2 Special semilinear and quasilinear equations 149
2.7.3 Divergent quasilinear and fully nonlinear equations 151
2.7.4 Quasilinear elliptic systems of orders 2 and $2m$ 157
2.7.5 Linearizing general divergent quasilinear and fully nonlinear systems 158

2.8 The Navier–Stokes equation 163
2.8.1 Introduction 163
2.8.2 The Stokes operator and saddle point problems 163
2.8.3 The Navier–Stokes operator and its linearization 167

PART II NUMERICAL METHODS

3 A general discretization theory 173
3.1 Introduction 173
3.2 Petrov–Galerkin and general discretization methods 175
3.3 Variational and classical consistency 185
3.4 Stability and consistency yield convergence 189
3.5 Techniques for proving stability 194
3.6 Stability implies invertibility 203
3.7 Solving nonlinear systems: Continuation and Newton’s method based upon the mesh independence principle (MIP) 205
3.7.1 Continuation methods 205
3.7.2 MIP for nonlinear systems 206

4 Conforming finite element methods (FEMs) 209
4.1 Introduction 209
4.2 Approximation theory for finite elements 212
4.2.1 Subdivisions and finite elements 212
4.2.2 Polynomial finite elements, triangular and rectangular $K$ 214
4.2.3 Interpolation in finite element spaces, an example 221
4.2.4 Interpolation errors and inverse estimates 229
4.2.5 Inverse estimates on nonquasiuniform triangulations 233
4.2.6 Smooth FEs on polyhedral domains, with O. Davydov 238
4.2.7 Curved boundaries 250
4.3 FEMs for linear problems 257
4.3.1 Finite element methods: a simple example, essential tools 258
4.3.2 Finite element methods for general linear equations and systems of orders 2 and 2m 264
4.3.3 General convergence theory for conforming FEMs 266
4.4 Finite element methods for divergent quasilinear elliptic equations and systems 273
4.5 General convergence theory for monotone and quasilinear operators 277
4.6 Mixed FEMs for Navier–Stokes and saddle point equations 281
  4.6.1 Navier–Stokes and saddle point equations 281
  4.6.2 Mixed FEMs for Stokes and saddle point equations 282
  4.6.3 Mixed FEMs for the Navier–Stokes operator 286
4.7 Variational methods for eigenvalue problems 288
  4.7.1 Introduction 288
  4.7.2 Theory for eigenvalue problems 289
  4.7.3 Different variational methods for eigenvalue problems 292
5 Nonconforming finite element methods 296
  5.1 Introduction 296
  5.2 Finite element methods for fully nonlinear elliptic problems 298
    5.2.1 Introduction 298
    5.2.2 Main ideas and results for the new FEM: An extended summary 299
    5.2.3 Fully nonlinear and general quasilinear elliptic equations 305
    5.2.4 Existence and convergence for semi-conforming FEMs 308
    5.2.5 Definition of nonconforming FEMs 311
    5.2.6 Consistency for nonconforming FEMs 317
    5.2.7 Stability for the linearized operator and convergence 319
    5.2.8 Discretization of equations and systems of order 2m 332
    5.2.9 Consistency, stability and convergence for m, q ≥ 1 336
    5.2.10 Numerical solution of the FE equations with Newton’s method 341
  5.3 FE and other methods for nonlinear boundary conditions 345
  5.4 Quadrature approximate FEMs 346
    5.4.1 Introduction 346
    5.4.2 Quadrature and cubature formulas 348
    5.4.3 Quadrature for second order linear problems 350
    5.4.4 Quadrature for second order fully nonlinear equations 357
    5.4.5 Quadrature FEMs for equations and systems of order 2m 361
    5.4.6 Two useful propositions 367
  5.5 Consistency, stability and convergence for FEMs with variational crimes 368
    5.5.1 Introduction 368
    5.5.2 Variational crimes for our standard example 370
5.5.3 FEMs with crimes for linear and quasilinear problems
5.5.4 Discrete coercivity and consistency
5.5.5 High order quadrature on edges
5.5.6 Violated boundary conditions
5.5.7 Violated continuity
5.5.8 Stability for nonconforming FEMs
5.5.9 Convergence, quadrature and solution of FEMs with crimes
5.5.10 Isoparametric FEMs

6 Adaptive finite element methods, by W. Dörfler
6.1 Introduction
6.1.1 The model problem
6.1.2 Singular solutions
6.1.3 A priori error bounds
6.1.4 Necessity of nonuniform mesh refinement
6.1.5 Optimal meshes – A heuristic argument
6.1.6 Optimal meshes for 2D corner singularities
6.1.7 The finite element method–Notation and requirements
6.2 The residual error estimator for the Poisson problem
6.2.1 Upper a posteriori bound
6.2.2 Lower a posteriori bound
6.2.3 The a posteriori error estimate
6.2.4 The adaptive finite element method
6.2.5 Stable refinement methods for triangulations in \( \mathbb{R}^2 \)
6.2.6 Convergence of the adaptive finite element method
6.2.7 Optimality
6.2.8 Other types of estimators
6.2.9 \( hp \) finite element method
6.3 Estimation of quantities of interest
6.3.1 Quantities of interest
6.3.2 Error estimates for point errors
6.3.3 Optimal meshes–A heuristic argument
6.3.4 The general approach

7 Discontinuous Galerkin methods (DCGMs), with V. Dolejší
7.1 Introduction
7.2 The model problem
7.3 Discretization of the problem
7.3.1 Triangulations
7.3.2 Broken Sobolev spaces
7.3.3 Extended variational formulation of the problem
7.3.4 Discretization
7.4 General linear elliptic problems
7.5 Semilinear and quasilinear elliptic problems 474
  7.5.1 Semilinear elliptic problems 474
  7.5.2 Variational formulation and discretization of the problem 475
  7.5.3 Quasilinear elliptic systems 477
  7.5.4 Discretization of the quasilinear systems 478

7.6 DCGMs are general discretization methods 482

7.7 Geometry of the mesh, error and inverse estimates 486
  7.7.1 Geometry of the mesh 487
  7.7.2 Inverse and interpolation error estimates 487

7.8 Penalty norms and consistency of the $J^c_h$ 491

7.9 Coercive linearized principal parts 494
  7.9.1 Coercivity of the original linearized principal parts 494
  7.9.2 Coercivity and boundedness in $V^h$ for the Laplacian 495
  7.9.3 Coercivity and boundedness in $V^h$ for the general linear and the semilinear case 499
  7.9.4 $V^h$-coercivity and boundedness for quasilinear problems 502

7.10 Consistency results for the $c^h, b^h, \ell^h$ 503
  7.10.1 Consistency of the $c^h$ and $b^h$ 503
  7.10.2 Consistency of the $\ell^h$ 505

7.11 Consistency properties of the $a^h$ 507
  7.11.1 Consistency of the $a^h$ for the Laplacian 507
  7.11.2 Consistency of the $a^h$ for general linear problems 511
  7.11.3 Consistency of the semilinear $a^h$ 514
  7.11.4 Consistency of the quasilinear $a^h$ for systems 518
  7.11.5 Consistency of the quasilinear $a^h$ for the equations of Houston, Robson, Sil, and for systems 523

7.12 Convergence for DCGMs 527

7.13 Solving nonlinear equations in DCGMs 532
  7.13.1 Introduction 532
  7.13.2 Discretized linearized quasilinear system and differentiable consistency 532

7.14 $hp$-variants of DCGM 538
  7.14.1 $hp$-finite element spaces 539
  7.14.2 $hp$-DCGMs 540
  7.14.3 $hp$-inverse and approximation error estimates 540
  7.14.4 Consistency and convergence of $hp$-DCGMs 542

7.15 Numerical experiences 546
  7.15.1 Scalar quasilinear equation 546
  7.15.2 System of the steady compressible Navier–Stokes equations 554

8 Finite difference methods 560
  8.1 Introduction 560
  8.2 Difference methods for simple examples, notation 562
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.3</td>
<td>Discrete Sobolev spaces</td>
<td>566</td>
</tr>
<tr>
<td>8.3.1</td>
<td>Notation and definitions</td>
<td>566</td>
</tr>
<tr>
<td>8.3.2</td>
<td>Discrete Sobolev spaces</td>
<td>569</td>
</tr>
<tr>
<td>8.4</td>
<td>General elliptic problems with Dirichlet conditions, and their</td>
<td></td>
</tr>
<tr>
<td></td>
<td>difference methods</td>
<td>572</td>
</tr>
<tr>
<td>8.4.1</td>
<td>General elliptic problems</td>
<td>572</td>
</tr>
<tr>
<td>8.4.2</td>
<td>Second order linear elliptic difference equations</td>
<td>574</td>
</tr>
<tr>
<td>8.4.3</td>
<td>Symmetric difference methods</td>
<td>581</td>
</tr>
<tr>
<td>8.4.4</td>
<td>Linear equations of order $2m$</td>
<td>583</td>
</tr>
<tr>
<td>8.4.5</td>
<td>Quasilinear elliptic equations of orders 2, and $2m$</td>
<td>584</td>
</tr>
<tr>
<td>8.4.6</td>
<td>Systems of linear and quasilinear elliptic equations</td>
<td>586</td>
</tr>
<tr>
<td>8.4.7</td>
<td>Fully nonlinear elliptic equations and systems</td>
<td>587</td>
</tr>
<tr>
<td>8.5</td>
<td>Convergence for difference methods</td>
<td>588</td>
</tr>
<tr>
<td>8.5.1</td>
<td>Discretization concepts in discrete Sobolev spaces</td>
<td>589</td>
</tr>
<tr>
<td>8.5.2</td>
<td>The operators $P_h, Q'_h$</td>
<td>591</td>
</tr>
<tr>
<td>8.5.3</td>
<td>Consistency for difference equations</td>
<td>594</td>
</tr>
<tr>
<td>8.5.4</td>
<td>$\gamma^h$-coercivity for linear(ized) elliptic difference equations</td>
<td>600</td>
</tr>
<tr>
<td>8.5.5</td>
<td>Stability and convergence for general elliptic difference equations</td>
<td>604</td>
</tr>
<tr>
<td>8.6</td>
<td>Natural boundary value problems of order 2</td>
<td>610</td>
</tr>
<tr>
<td>8.6.1</td>
<td>Analysis for natural boundary value problems</td>
<td>611</td>
</tr>
<tr>
<td>8.6.2</td>
<td>Difference methods for natural boundary value problems</td>
<td>613</td>
</tr>
<tr>
<td>8.7</td>
<td>Other difference methods on curved boundaries</td>
<td>622</td>
</tr>
<tr>
<td>8.7.1</td>
<td>The Shortley–Weller–Collatz method for linear equations</td>
<td>623</td>
</tr>
<tr>
<td>8.8</td>
<td>Asymptotic expansions, extrapolation, and defect corrections</td>
<td>626</td>
</tr>
<tr>
<td>8.8.1</td>
<td>A difference method based on polynomial interpolation for linear, and semilinear equations</td>
<td>627</td>
</tr>
<tr>
<td>8.8.2</td>
<td>Asymptotic expansions for other methods</td>
<td>630</td>
</tr>
<tr>
<td>8.9</td>
<td>Numerical experiments for the von Kármán equations, with C.S. Chien</td>
<td>633</td>
</tr>
<tr>
<td>9</td>
<td>Variational methods for wavelets, with S. Dahlke</td>
<td>635</td>
</tr>
<tr>
<td>9.1</td>
<td>Introduction</td>
<td>635</td>
</tr>
<tr>
<td>9.2</td>
<td>The scope of problems</td>
<td>637</td>
</tr>
<tr>
<td>9.3</td>
<td>Wavelet analysis</td>
<td>639</td>
</tr>
<tr>
<td>9.3.1</td>
<td>The discrete wavelet transform</td>
<td>640</td>
</tr>
<tr>
<td>9.3.2</td>
<td>Biorthogonal bases</td>
<td>644</td>
</tr>
<tr>
<td>9.3.3</td>
<td>Wavelets and function spaces</td>
<td>646</td>
</tr>
<tr>
<td>9.3.4</td>
<td>Wavelets on domains</td>
<td>647</td>
</tr>
<tr>
<td>9.3.5</td>
<td>Evaluation of nonlinear functionals</td>
<td>652</td>
</tr>
<tr>
<td>9.4</td>
<td>Stable discretizations and preconditioning</td>
<td>653</td>
</tr>
<tr>
<td>9.5</td>
<td>Applications to elliptic equations</td>
<td>659</td>
</tr>
<tr>
<td>9.6</td>
<td>Saddle point and (Navier–)Stokes equations</td>
<td>664</td>
</tr>
<tr>
<td>9.6.1</td>
<td>Saddle point equations</td>
<td>664</td>
</tr>
<tr>
<td>9.6.2</td>
<td>Navier–Stokes equations</td>
<td>666</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------</td>
<td>------</td>
</tr>
<tr>
<td>9.7</td>
<td>Adaptive wavelet methods, by T. Raasch</td>
<td>669</td>
</tr>
<tr>
<td>9.7.1</td>
<td>Nonlinear approximation with wavelet systems</td>
<td>672</td>
</tr>
<tr>
<td>9.7.2</td>
<td>Wavelet matrix compression</td>
<td>675</td>
</tr>
<tr>
<td>9.7.3</td>
<td>Adaptive wavelet–Galerkin methods</td>
<td>678</td>
</tr>
<tr>
<td>9.7.4</td>
<td>Adaptive descent iterations</td>
<td>680</td>
</tr>
<tr>
<td>9.7.5</td>
<td>Nonlinear stationary problems</td>
<td>683</td>
</tr>
</tbody>
</table>

Bibliography 686

Index 733