Alexander Soifer

The Mathematical Coloring Book

Mathematics of Coloring and the Colorful Life of its Creators

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Springer
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