Symplectic Geometry

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Symplectic geometry is the mathematical apparatus of such areas of physics as classical mechanics, geometrical optics and thermodynamics. Whenever the equations of a theory can be gotten out of a variational principle, symplectic geometry clears up and systematizes the relations between the quantities entering into the theory. Symplectic geometry simplifies and makes perceptible the frightening formal apparatus of Hamiltonian dynamics and the calculus of variations in the same way that the ordinary geometry of vector spaces reduces cumbersome coordinate computations to a small number of simple basic principles.

In the present survey the simplest fundamental concepts of symplectic geometry are expounded. The applications of symplectic geometry to mechanics are discussed in greater detail in volume 3 of this series, and its applications to the theory of integrable systems and to quantization receive more thorough review in the articles of A.A. Kirillov and of B.A. Dubrovin, I.M. Krichever and S.P. Novikov in this volume.

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