CONTENTS

Preface

An Introduction to Enumeration

Section 1 Elementary Counting Principles
What Is Combinatorics? 1; The Sum Principle 1; The Product Principle 3; Ordered Pairs 4; Cartesian Product of Sets 4; The General Form of the Product Principle 4; Lists with Distinct Elements 5; Lists with Repeats Allowed 7; Stirling's Approximation for n! 7; EXERCISES 8

Section 2 Functions and the Pigeonhole Principle
Functions 14; Relations 15; Definition of Function 15; The Number of Functions 16; One-to-One Functions 16; Onto Functions and Bijections 18; The Extended Pigeonhole Principle 20; Ramsey Numbers 21; *Using Functions to Describe Ramsey Numbers 22; EXERCISES 23

Section 3 Subsets
The Number of Subsets of a Set 27; Binomial Coefficients 28; k-Element Subsets 29; Labelings with Two Labels 29; Pascal’s Triangle 30; How Fast Does the Number of Subsets Grow? 33; Recursion and Iteration 34; EXERCISES 35

Section 4 Using Binomial Coefficients
The Binomial Theorem 40; Multinomial Coefficients 43; The Multinomial Theorem 45; Multinomial Coefficients from Binomial Coefficients 46; Lattice Paths 46; EXERCISES 50

Section 5 Mathematical Induction
The Principle of Induction 54; Proving That Formulas Work 54; Informal Induction Proofs 56; Inductive Definition 56; The
2 Equivalence Relations, Partitions, and Multisets

Section 1 Equivalence Relations
The Idea of Equivalence 69; Equivalence Relations 70; Circular Arrangements 70; Equivalence Classes 72; Counting Equivalence Classes 73; The Inverse Image Relation 74; The Number of Partitions with Specified Class Sizes 77; EXERCISES 79

Section 2 Distributions and Multisets
The Idea of a Distribution 83; Ordered Distributions 87; Distributing Identical Objects to Distinct Recipients 89; Ordered Compositions 92; Multisets 93; Broken Permutations of a Set 94; EXERCISES 96

Section 3 Partitions and Stirling Numbers
Partitions of an m-Element Set into n Classes 99; Stirling's Triangle of the Second Kind 99; The Inverse Image Partition of a Function 100; Onto Functions and Stirling Numbers 101; Stirling Numbers of the First Kind 101; Stirling Numbers of the Second Kind as Polynomial Coefficients 102; Stirling's Triangle of the First Kind 104; The Total Number of Partitions of a Set 105; EXERCISES 106

Section 4 Partitions of Integers
Distributing Identical Objects to Identical Recipients 110; Type Vector of a Partition and Decreasing Lists 110; The Number of Partitions of m into n Parts 111; Ferrers Diagrams 112; Conjugate Partitions 112; The Total Number of Partitions of m 114; EXERCISES 115

Suggested Reading for Chapter 2 118
Algebraic Counting Techniques

Section 1  The Principle of Inclusion and Exclusion
The Size of a Union of Three Overlapping Sets 119; The Number of Onto Functions 120; Counting Arrangements with or without Certain Properties 122; The Basic Counting Functions $\mathbb{N}_\geq$ and $\mathbb{N}_=$ 123; The Principle of Inclusion and Exclusion 124; Onto Functions and Stirling Numbers 126; Examples of Using the Principle of Inclusion and Exclusion 127; Derangements 131; *Level Sums and Inclusion–Exclusion Counting 131; *Examples of Level Sum Inclusion and Exclusion 133; EXERCISES 134

Section 2  The Concept of a Generating Function
Symbolic Series 138; Power Series 143; What Is a Generating Function? 144; The Product Principle for Generating Functions 145; The Generating Function for Multisets 146; Polynomial Generating Functions 147; Extending the Definition of Binomial Coefficients 148; The Extended Binomial Theorem 148; EXERCISES 149

Section 3  Applications to Partitions and Inclusion–Exclusion
Pólya’s Change-Making Example 154; Systems of Linear Recurrences from Products of Geometric Series 155; Generating Functions for Integer Partitions 158; Generating Functions Sometimes Replace Inclusion–Exclusion 162; *Generating Functions and Inclusion–Exclusion on Level Sums 163; EXERCISES 165

Section 4  Recurrence Relations and Generating Functions
The Idea of a Recurrence Relation 169; How Generating Functions Are Relevant 170; Second-Order Linear Recurrence Relations 172; The Original Fibonacci Problem 177; General Techniques 178; EXERCISES 180

Section 5  Exponential Generating Functions
Indicator Functions 184; Exponential Generating Functions 184; Products of Exponential Generating Functions 185; The Exponential Generating Function for Onto Functions 188; The Product Principle for Exponential Generating Functions 189; Putting Lists Together and Preserving Order 190; Exponential Generating Functions for Words 192; Solving Recurrence Relations with
4 Graph Theory

Section 1 Eulerian Walks and the Idea of Graphs
The Concept of a Graph 200; Multigraphs and the Königsberg Bridge Problem 202; Walks, Paths, and Connectivity 204; Eulerian Graphs 206; EXERCISES 208

Section 2 Trees
The Chemical Origins of Trees 211; Basic Facts about Trees 212; Spanning Trees 215; *The Number of Trees 217; EXERCISES 222

Section 3 Shortest Paths and Search Trees
Rooted Trees 226; Breadth-First Search Trees 228; Shortest Path Spanning Trees 229; Bridges 231; Depth-First Search 232; Depth-First Numbering 233; Finding Bridges 234; *An Efficient Bridge-Finding Algorithm for Computers 235; Backtracking 235; Decision Graphs 237; EXERCISES 238

Section 4 Isomorphism and Planarity
The Concept of Isomorphism 242; Checking Whether Two Graphs Are Isomorphic 243; Planarity 245; Euler’s Formula 246; An Inequality to Check for Nonplanarity 246; EXERCISES 249

Section 5 Digraphs
Directed Graphs 252; Walks and Connectivity 253; Tournament Digraphs 253; Hamiltonian Paths 254; Transitive Closure 255; Reachability 256; Modifying Breadth-First Search for Strict Reachability 257; Orientable Graphs 258; Graphs without Bridges Are Orientable 258; EXERCISES 259

Section 6 Coloring
The Four-Color Theorem 263; Chromatic Number 263; Maps and Duals 264; The Five-Color Theorem 265; Kempe’s Attempted Proof 268; Using Backtracking to Find a Coloring 269; EXERCISES 272
Section 7 Graphs and Matrices
Adjacency Matrix of a Graph 277; Matrix Powers and Walks 277; Connectivity and Transitive Closure 279; Boolean Operations 280; *The Matrix–Tree Theorem 281; *The Number of Eulerian Walks in a Digraph 284; EXERCISES 285

Suggested Reading for Chapter 4 290

5 Matching and Optimization

Section 1 Matching Theory
The Idea of Matching 291; Making a Bigger Matching 294; A Procedure for Finding Alternating Paths in Bipartite Graphs 295; Constructing Bigger Matchings 296; Testing for Maximum-Sized Matchings by Means of Vertex Covers 297; Hall’s “Marriage” Theorem 299; Term Rank and Line Covers of Matrices 300; Permutation Matrices and the Birkhoff–von Neumann Theorem 301; *Finding Alternating Paths in Nonbipartite Graphs 302; EXERCISES 308

Section 2 The Greedy Algorithm
The SDR Problem with Representatives That Cost Money 314; The Greedy Method 314; The Greedy Algorithm 316; Matroids Make the Greedy Algorithm Work 316; How Much Time Does the Algorithm Take? 318; The Greedy Algorithm and Minimum-Cost Independent Sets 320; The Forest Matroid of a Graph 321; Minimum-Cost Spanning Trees 322; EXERCISES 324

Section 3 Network Flows
Transportation Networks 328; The Concept of Flow 328; Cuts in Networks 330; Flow-Augmenting Paths 332; The Labeling Algorithm for Finding Flow-Augmenting Paths 333; The Max-Flow Min-Cut Theorem 336; *More Efficient Algorithms 337; EXERCISES 340

Section 4 Flows, Connectivity, and Matching
Connectivity and Menger’s Theorem 345; Flows, Matchings, and Systems of Distinct Representatives 348; Minimum-Cost SDRs 350; Minimum-Cost Matchings and Flows with Edge Costs 352; The Potential Algorithm for Finding Minimum-Cost
6 Combinatorial Designs

Section 1 Latin Squares and Graeco–Latin Squares
How Latin Squares Are Used 359; Randomization for Statistical Purposes 360; Orthogonal Latin Squares 362; Euler’s 36-Officers Problem 363; Congruence Modulo an Integer $n$ 363; Using Arithmetic Modulo $n$ to Construct Latin Squares 365; Orthogonality and Arithmetic Modulo $n$ 367; Compositions of Orthogonal Latin Squares 368; Orthogonal Arrays and Latin Squares 372; *The Construction of a 10 by 10 Graeco–Latin Square 375; EXERCISES 377

Section 2 Block Designs
How Block Designs Are Used 379; Basic Relationships among the Parameters 380; The Incidence Matrix of a Design 381; An Example of a BIBD 383; Isomorphism of Designs 383; The Dual of a Design 384; Symmetric Designs 385; The Necessary Conditions Need Not Be Sufficient 387; EXERCISES 388

Section 3 Construction and Resolvability of Designs
A Problem That Requires a Big Design 391; Cyclic Designs 391; Resolvable Designs 395; $\infty$-Cyclic Designs 396; Triple Systems 397; Kirkman’s Schoolgirl Problem 398; Constructing New Designs from Old 399; Complementary Designs 400; Unions of Designs 401; Product Designs 402; Composition of Designs 402; *The Construction of Kirkman Triple Systems 403; EXERCISES 406

Section 4 Affine and Projective Planes
Affine Planes 409; Postulates for Affine Planes 412; The Concept of a Projective Plane 414; Basic Facts about Projective Planes 417; Projective Planes and Block Designs 418; Planes and Resolvable Designs 419; Planes and Orthogonal Latin Squares 420; EXERCISES 422
Section 5  Codes and Designs
The Concept of an Error-Correcting Code 424; Hamming Distance 425; Perfect Codes 427; Linear Codes 429; The Hamming Codes 431; Constructing Designs from Codes 433; Highly Balanced Designs 434; t-Designs 435; Codes and Latin Squares 436; EXERCISES 438

Suggested Reading for Chapter 6

7  Ordered Sets

Section 1  Partial Orderings
What Is an Ordering? 442; Linear Orderings 444; Maximal and Minimal Elements 445; The Diagram and Covering Graph 446; Ordered Sets as Transitive Closures of Digraphs 449; Trees as Ordered Sets 449; Weak Orderings 450; Interval Orders 451; EXERCISES 454

Section 2  Linear Extensions and Chains
The Idea of a Linear Extension 458; Dimension of an Ordered Set 459; Topological Sorting Algorithms 460; Chains in Ordered Sets 460; Chain Decompositions of Posets 462; Finding Chain Decompositions 464; Alternating Walks 467; Finding Alternating Walks 470; EXERCISES 471

Section 3  Lattices
What Is a Lattice? 477; The Partition Lattice 479; The Bond Lattice of a Graph 481; The Algebraic Description of Lattices 482; EXERCISES 484

Section 4  Boolean Algebras
The Idea of a Complement 488; Boolean Algebras 490; Boolean Algebras of Statements 491; Combinatorial Gate Networks 494; Boolean Polynomials 496; DeMorgan's Laws 496; Disjunctive Normal Form 497; All Finite Boolean Algebras Are Subset Lattices 499; EXERCISES 501

Section 5  Möbius Functions
A Review of Inclusion and Exclusion 505; The Zeta Matrix 506; The Möbius Matrix 508; The Möbius Function 509; Equations
Contents

That Describe the Möbius Function 511; The Number-Theoretic Möbius Function 513; The Number of Connected Graphs 515; A General Method of Computing Möbius Functions of Lattices 517; The Möbius Function of the Partition Lattice 518; EXERCISES 519

Section 6 Products of Orderings
The Idea of a Product 523; Products of Ordered Sets and Möbius Functions 524; Products of Ordered Sets and Dimension 526; Width and Dimension of Ordered Sets 528; EXERCISES 529

Suggested Reading for Chapter 7

8 Enumeration under Group Action

Section 1 Permutation Groups
Permutations and Equivalence Relations 532; The Group Properties 535; Powers of Permutations 537; Permutation Groups 538; Associating a Permutation with a Geometric Motion 539; Abstract Groups 540; EXERCISES 542

Section 2 Groups Acting on Sets
Groups Acting on Sets 545; Orbits as Equivalence Classes 547; The Subgroup Fixing a Point and Cosets 548; The Size of a Subgroup 551; The Subgroup Generated by a Set 552; The Cycles of a Permutation 553; EXERCISES 557

Section 3 Pólya's Enumeration Theorem
The Cauchy–Frobenius–Burnside Theorem 562; Enumerators of Colorings 564; Generating Functions for Orbits 566; Using the Orbit–Fixed Point Lemma 568; Orbits of Functions 570; The Orbit Enumerator for Functions 572; How Cycle Structure Interacts with Colorings 573; The Pólya–Redfield Theorem 575; EXERCISES 578

Suggested Reading for Chapter 8

Answers to Exercises

Index