## Contents

**Preface** vii

**Introduction** 1

- What is in the Book 1
- How to Use the Book 3
- Notation and Conventions 5

**VII THEORIES OF RECURSIVE FUNCTIONS** 9

**VII.1 Measures of Complexity** 10

- Static complexity measures * 11
- Shortening proofs by adding axioms * 17
- Definition of a dynamic complexity measure 19
- First properties of dynamic complexity measures 23

**VII.2 Speed of Computations** 26

- Upper and lower bounds of the complexities of a function 26
- The Compression Theorem * 32
- Functions with best complexity 34
- The Speed-Up Theorem 38

**VII.3 Complexity Classes** 47

- Hierarchies for the recursive functions: successor levels * 50
- Hierarchies for the recursive functions: limit levels 55
- Hierarchies for the recursive functions: exhaustiveness * 59
- Names for complexity classes * 61

**VII.4 Time and Space Measures** 67

- One-tape Turing machines 67
- Other Turing machine models * 70
- Linear Speed-Up 76
- Hierarchy theorems 80
- Space versus time 84
- Nondeterministic Turing machines 86
## VIII HIERARCHIES OF RECURSIVE FUNCTIONS

### VIII.1 Small Time and Space Bounds

- Real time ........................................ 147
- Constant space ................................ 150
- Logarithmic space .............................. 160

### VIII.2 Deterministic Polynomial Time

- Polynomial time computable functions ........ 163
- Closure properties .............................. 164
- Alternative characterizations ................. 170
- Rate of growth .................................. 177
- Feasibly computable functions ................. 178
- The class P ...................................... 181
- Logarithmic space again and beyond .......... 184
- A look inside P .................................. 187
- Polynomial time degrees ......................... 188

### VIII.3 Nondeterministic Polynomial Time

- The class NP ...................................... 198
- Deterministic polynomial time again .......... 198
- NP sets as analogues of r.e. sets: successes . 199
- NP sets as analogues of r.e. sets: failures .. 208
- Relativizations .................................. 213
- Polynomial time degrees again ................ 222

### VIII.4 The Polynomial Time Hierarchy

- Truth in Bounded Quantifier Arithmetic ...... 224
- Types of bounded quantifiers .................. 225
- The Polynomial Time Hierarchy ................ 226
- Second-Order Logic on finite domains ......... 227
- The levels of the Polynomial Time Hierarchy .... 228
- Relativizations .................................. 231
IX RECURSIVELY ENUMERABLE SETS 357

IX.1 Global Properties of Recursive Sets 358
   Characterizations of the lattice of recursive sets 358
   The complexity of the theory of recursive sets 361
   Homogeneity 361
   Automorphisms 362
   Absolute definability 363
   Ultrafilters and models of fragments of Arithmetic * 364

IX.2 Local Properties of R.E. Sets 368
   Splitting theorems 368
   Hyperhypersimple sets 374
   R-maximal sets 376
   Maximal sets 379
   Sets without maximal supersets 385
   The world of simple sets * 393

IX.3 Global Properties of R.E. Sets 401
   The complexity of the theory of r.e. sets * 401
   Absolute definability 403
   Homogeneity 411
   Automorphisms 414
   Orbits * 422
   Definable and invariant classes of r.e. degrees * 423

IX.4 Complexity of R.E. Sets 424
   Nonspeedable sets 424
   Effectively speedable sets 431
   Existence theorems 435
   Complexity sequences * 438

IX.5 Inductive Inference of R.E. Sets * 447
   Identification by explanation of sets 447
   Identification by explanation of partial functions 452

X RECURSIVELY ENUMERABLE DEGREES 455

X.1 The Finite Injury Priority Method 456
   Motivation 456
   Embeddability results 459
   Sacks agreement method 464
   A tactical variation * 467

X.2 Effective Baire Category * 467
   Categorical formulation of finite injury arguments 468
   The permitting method 470
   Degrees r.e. in smaller degrees * 471
<table>
<thead>
<tr>
<th>Chapter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inside truth-table degrees</td>
</tr>
<tr>
<td>Inside weak truth-table degrees</td>
</tr>
<tr>
<td>Inside Turing degrees</td>
</tr>
<tr>
<td><strong>X.9</strong> Index Sets</td>
</tr>
<tr>
<td>Complexity of index sets</td>
</tr>
<tr>
<td>Specific index sets</td>
</tr>
<tr>
<td>Applications of index sets to degrees</td>
</tr>
<tr>
<td>Global structure</td>
</tr>
</tbody>
</table>

**XI LIMIT SETS**

<table>
<thead>
<tr>
<th>Section</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>XI.1</strong> Jump Classes</td>
</tr>
<tr>
<td>Domination properties</td>
</tr>
<tr>
<td>Jump classes</td>
</tr>
<tr>
<td>Hops</td>
</tr>
<tr>
<td>Jump inversion below 0'</td>
</tr>
<tr>
<td>Generalized jump classes</td>
</tr>
<tr>
<td><strong>XI.2</strong> 1-Generic Degrees</td>
</tr>
<tr>
<td>Full approximation arguments</td>
</tr>
<tr>
<td>Permitting below r.e. degrees</td>
</tr>
<tr>
<td>Permitting below GL2 degrees</td>
</tr>
<tr>
<td>Permitting below 1-generic degrees below 0'</td>
</tr>
<tr>
<td><strong>XI.3</strong> Structure Theory</td>
</tr>
<tr>
<td>The Diamond Theorem</td>
</tr>
<tr>
<td>Incomparable degrees</td>
</tr>
<tr>
<td>The Capping Theorem</td>
</tr>
<tr>
<td>The Cupping Theorem</td>
</tr>
<tr>
<td>The Complementation Theorem</td>
</tr>
<tr>
<td>Exact pairs and ideals</td>
</tr>
<tr>
<td><strong>XI.4</strong> Minimal Degrees</td>
</tr>
<tr>
<td>Methodology</td>
</tr>
<tr>
<td>Minimal degrees below 0'</td>
</tr>
<tr>
<td>Full approximation arguments</td>
</tr>
<tr>
<td>Permitting below r.e. degrees</td>
</tr>
<tr>
<td>The initial segments of the degrees below 0'</td>
</tr>
<tr>
<td><strong>XI.5</strong> Global Properties</td>
</tr>
<tr>
<td>Definability from parameters</td>
</tr>
<tr>
<td>The complexity of the theory of degrees below 0'</td>
</tr>
<tr>
<td>Absolute definability</td>
</tr>
<tr>
<td>Homogeneity</td>
</tr>
<tr>
<td>Automorphisms</td>
</tr>
<tr>
<td>Subclasses of degrees below 0'</td>
</tr>
<tr>
<td>Definability of 0' in the degrees</td>
</tr>
</tbody>
</table>
## Contents

<table>
<thead>
<tr>
<th>XXI.6 Many-One Degrees *</th>
<th>729</th>
</tr>
</thead>
<tbody>
<tr>
<td>Partial many-one reducibility</td>
<td>729</td>
</tr>
<tr>
<td>The mini-jump operator</td>
<td>732</td>
</tr>
<tr>
<td>$\Delta^0_2$ many-one degrees</td>
<td>734</td>
</tr>
</tbody>
</table>

### XII ARITHMETICAL SETS

#### XII.1 Forcing in Arithmetic
- Definition of forcing                     | 737 |
- Generic sets                              | 738 |
- Genericity without forcing *              | 741 |
- An alternative approach to forcing *      | 744 |
- History of the notion of forcing *        | 747 |
- Product forcing *                         | 748 |
- Local forcing on trees                    | 751 |

#### XII.2 Applications of Forcing
- Turing degrees                            | 753 |
- Arithmetical reducibilities *             | 755 |
- Arithmetical definability *              | 758 |
- Implicit arithmetical definability *      | 760 |
- $\omega$-Hops                             | 765 |

#### XII.3 Turing Degrees of Arithmetical Sets
- Local and global properties               | 769 |
- Cones of minimal covers                   | 773 |
- Definability of the Turing degrees of arithmetical sets | 776 |

### XIII ARITHMETICAL DEGREES

#### XIII.1 The Theory of Arithmetical Degrees
- The finite extension method               | 781 |
- The tree method                           | 784 |
- Arithmetical jump                         | 787 |
- Global properties                         | 787 |
- Arithmetical degrees below $0'_a$          | 791 |

#### XIII.2 An Analogue of R.E. Sets
- Basic properties                          | 794 |
- Representation of infinite hops           | 796 |
- The complexity of $\omega$-r.e.a. sets    | 797 |

#### XIII.3 An Analogue of Post’s Problem
- A first approximation to the construction | 798 |
- The real construction                     | 800 |
- The basic module of the construction      | 804 |
- The ingredients of the construction       | 806 |
- An analogue of the Friedberg-Muchnik Theorem | 809 |
**Contents**

XIII.4 An Analogue of the Jump Classes .......................... 811
   The main result ........................................ 811
   Jump classes ........................................... 815
   Jump inversion ......................................... 816
XIII.5 Comparison with the R.E. Degrees .......................... 817
   Minimal pairs ........................................... 817
   The Diamond Theorem .................................... 824

XIV ENUMERATION DEGREES ........................................ 827
XIV.1 Enumeration Degrees ....................................... 827
   Total degrees ......................................... 829
XIV.2 The Theory of Enumeration Degrees ........................ 833
   Minimal pairs ........................................... 834
   Minimal covers ......................................... 836
   Global properties ....................................... 842
XIV.3 Enumeration Degrees below $0'_{e}$ .......................... 847
   Density ................................................... 849
   Greatest lower bounds .................................. 851
   Least upper bounds ...................................... 853
   Lattice embeddings ...................................... 853
   Global properties ....................................... 856
XIV.4 A Model of the Lambda Calculus * ........................... 857

Bibliography .................................................. 863

Notation Index ............................................... 923

Subject Index ............................................... 929