Discrete Integrable Geometry and Physics

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CLARENDON PRESS • OXFORD
1999
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*The colour plates were prepared by Tim Hoffmann, and can be found between pages 100 and 101.*

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