Bruce P. Palka

An Introduction to Complex Function Theory

With 138 Illustrations

Springer
## Contents

Preface vii

I The Complex Number System 1

1 The Algebra and Geometry of Complex Numbers 1
   1.1 The Field of Complex Numbers 1
   1.2 Conjugate, Modulus, and Argument 5

2 Exponentials and Logarithms of Complex Numbers 13
   2.1 Raising $e$ to Complex Powers 13
   2.2 Logarithms of Complex Numbers 15
   2.3 Raising Complex Numbers to Complex Powers 16

3 Functions of a Complex Variable 17
   3.1 Complex Functions 17
   3.2 Combining Functions 19
   3.3 Functions as Mappings 20

4 Exercises for Chapter I 25

II The Rudiments of Plane Topology 33

1 Basic Notation and Terminology 33
   1.1 Disks 33
   1.2 Interior Points, Open Sets 34
   1.3 Closed Sets 34
   1.4 Boundary, Closure, Interior 35
   1.5 Sequences 35
   1.6 Convergence of Complex Sequences 36
   1.7 Accumulation Points of Complex Sequences 37

2 Continuity and Limits of Functions 39
   2.1 Continuity 39
   2.2 Limits of Functions 43

3 Connected Sets 47
### Contents

#### III Analytic Functions

1 Complex Derivatives .......................... 62
   1.1 Differentiability .......................... 62
   1.2 Differentiation Rules ...................... 64
   1.3 Analytic Functions ......................... 67

2 The Cauchy-Riemann Equations ............... 68
   2.1 The Cauchy-Riemann System of Equations .... 68
   2.2 Consequences of the Cauchy-Riemann Relations .. 73

3 Exponential and Trigonometric Functions ..... 75
   3.1 Entire Functions ........................... 75
   3.2 Trigonometric Functions .................... 77
   3.3 The Principal Arccsine and Arctangent Functions .. 81

4 Branches of Inverse Functions ............... 85
   4.1 Branches of Inverse Functions .............. 85
   4.2 Branches of the \( p \)th-root Function ........ 87
   4.3 Branches of the Logarithm Function .......... 91
   4.4 Branches of the \( \lambda \)-power Function ... 92

5 Differentiability in the Real Sense .......... 96
   5.1 Real Differentiability ...................... 96
   5.2 The Functions \( f_1 \) and \( f_2 \) ................. 98

6 Exercises for Chapter III ..................... 101

#### IV Complex Integration

1 Paths in the Complex Plane .................. 109
   1.1 Paths .................................. 109
   1.2 Smooth and Piecewise Smooth Paths ...... 112
   1.3 Parametrizing Line Segments ............... 114
   1.4 Reverse Paths, Path Sums ................. 115
Contents

7.2 Contractible Paths ........................................ 203
8 Exercises for Chapter V ................................. 204

VI Harmonic Functions ........................................ 214
1 Harmonic Functions .......................................... 215
  1.1 Harmonic Conjugates .................................. 215
2 The Mean Value Property .................................. 219
  2.1 The Mean Value Property .............................. 219
  2.2 Functions Harmonic in Annuli ........................ 221
3 The Dirichlet Problem for a Disk ........................ 226
  3.1 A Heat Flow Problem .................................. 226
  3.2 Poisson Integrals ..................................... 228
4 Exercises for Chapter VI .................................... 238

VII Sequences and Series of Analytic Functions .............. 243
1 Sequences of Functions ..................................... 243
  1.1 Uniform Convergence .................................. 243
  1.2 Normal Convergence ................................... 246
2 Infinite Series ............................................. 248
  2.1 Complex Series ....................................... 248
  2.2 Series of Functions .................................... 253
3 Sequences and Series of Analytic Functions ............... 256
  3.1 General Results ....................................... 256
  3.2 Limit Superior of a Sequence ......................... 259
  3.3 Taylor Series ......................................... 260
  3.4 Laurent Series ....................................... 269
4 Normal Families ............................................ 278
  4.1 Normal Subfamilies of $C(U)$ ......................... 278
  4.2 Equicontinuity ....................................... 279
  4.3 The Arzelà-Ascoli and Montel Theorems ............... 282
5 Exercises for Chapter VII ................................ 286

VIII Isolated Singularities of Analytic Functions ............ 300
1 Zeros of Analytic Functions ................................ 300
  1.1 The Factor Theorem for Analytic Functions .......... 300
  1.2 Multiplicity .......................................... 303
  1.3 Discrete Sets, Discrete Mappings .................... 306
2 Isolated Singularities ...................................... 309
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Definition and Classification of Isolated Singularities</td>
<td>309</td>
</tr>
<tr>
<td>2.2</td>
<td>Removable Singularities</td>
<td>310</td>
</tr>
<tr>
<td>2.3</td>
<td>Poles</td>
<td>311</td>
</tr>
<tr>
<td>2.4</td>
<td>Meromorphic Functions</td>
<td>318</td>
</tr>
<tr>
<td>2.5</td>
<td>Essential Singularities</td>
<td>319</td>
</tr>
<tr>
<td>2.6</td>
<td>Isolated Singularities at Infinity</td>
<td>322</td>
</tr>
<tr>
<td>3</td>
<td>The Residue Theorem and its Consequences</td>
<td>323</td>
</tr>
<tr>
<td>3.1</td>
<td>The Residue Theorem</td>
<td>323</td>
</tr>
<tr>
<td>3.2</td>
<td>Evaluating Integrals with the Residue Theorem</td>
<td>326</td>
</tr>
<tr>
<td>3.3</td>
<td>Consequences of the Residue Theorem</td>
<td>339</td>
</tr>
<tr>
<td>4</td>
<td>Function Theory on the Extended Plane</td>
<td>349</td>
</tr>
<tr>
<td>4.1</td>
<td>The Extended Complex Plane</td>
<td>349</td>
</tr>
<tr>
<td>4.2</td>
<td>The Extended Plane and Stereographic Projection</td>
<td>350</td>
</tr>
<tr>
<td>4.3</td>
<td>Functions in the Extended Setting</td>
<td>352</td>
</tr>
<tr>
<td>4.4</td>
<td>Topology in the Extended Plane</td>
<td>354</td>
</tr>
<tr>
<td>4.5</td>
<td>Meromorphic Functions and the Extended Plane</td>
<td>356</td>
</tr>
<tr>
<td>5</td>
<td>Exercises for Chapter VIII</td>
<td>362</td>
</tr>
</tbody>
</table>

**IX** Conformal Mapping 374

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Conformal Mappings</td>
<td>375</td>
</tr>
<tr>
<td>1.1</td>
<td>Curvilinear Angles</td>
<td>375</td>
</tr>
<tr>
<td>1.2</td>
<td>Diffeomorphisms</td>
<td>377</td>
</tr>
<tr>
<td>1.3</td>
<td>Conformal Mappings</td>
<td>379</td>
</tr>
<tr>
<td>1.4</td>
<td>Some Standard Conformal Mappings</td>
<td>383</td>
</tr>
<tr>
<td>1.5</td>
<td>Self-Mappings of the Plane and Unit Disk</td>
<td>388</td>
</tr>
<tr>
<td>1.6</td>
<td>Conformal Mappings in the Extended Plane</td>
<td>389</td>
</tr>
<tr>
<td>2</td>
<td>Möbius Transformations</td>
<td>391</td>
</tr>
<tr>
<td>2.1</td>
<td>Elementary Möbius Transformations</td>
<td>391</td>
</tr>
<tr>
<td>2.2</td>
<td>Möbius Transformations and Matrices</td>
<td>392</td>
</tr>
<tr>
<td>2.3</td>
<td>Fixed Points</td>
<td>394</td>
</tr>
<tr>
<td>2.4</td>
<td>Cross-ratios</td>
<td>396</td>
</tr>
<tr>
<td>2.5</td>
<td>Circles in the Extended Plane</td>
<td>398</td>
</tr>
<tr>
<td>2.6</td>
<td>Reflection and Symmetry</td>
<td>399</td>
</tr>
<tr>
<td>2.7</td>
<td>Classification of Möbius Transformations</td>
<td>402</td>
</tr>
<tr>
<td>2.8</td>
<td>Invariant Circles</td>
<td>408</td>
</tr>
<tr>
<td>3</td>
<td>Riemann's Mapping Theorem</td>
<td>416</td>
</tr>
<tr>
<td>3.1</td>
<td>Preparations</td>
<td>416</td>
</tr>
<tr>
<td>3.2</td>
<td>The Mapping Theorem</td>
<td>419</td>
</tr>
<tr>
<td>4</td>
<td>The Carathéodory-Osgood Theorem</td>
<td>423</td>
</tr>
<tr>
<td>4.1</td>
<td>Topological Preliminaries</td>
<td>423</td>
</tr>
<tr>
<td>4.2</td>
<td>Double Integrals</td>
<td>426</td>
</tr>
</tbody>
</table>
4.3 Conformal Modulus ........................................... 427
4.4 Extending Conformal Mappings of the Unit Disk ... 440
4.5 Jordan Domains ............................................. 445
4.6 Oriented Boundaries ....................................... 447

5 Conformal Mappings onto Polygons ...................... 450
5.1 Polygons .................................................. 450
5.2 The Reflection Principle ................................ 451
5.3 The Schwarz-Christoffel Formula ....................... 454

6 Exercises for Chapter IX ................................... 466

X Constructing Analytic Functions .......................... 477
1 The Theorem of Mittag-Leffler .............................. 477
1.1 Series of Meromorphic Functions ....................... 477
1.2 Constructing Meromorphic Functions .................... 479
1.3 The Weierstrass $\wp$-function ......................... 486
2 The Theorem of Weierstrass ................................ 490
2.1 Infinite Products ......................................... 490
2.2 Infinite Products of Functions ......................... 493
2.3 Infinite Products and Analytic Functions ............ 495
2.4 The Gamma Function ...................................... 504

3 Analytic Continuation ..................................... 507
3.1 Extending Functions by Means of Taylor Series ....... 507
3.2 Analytic Continuation .................................... 510
3.3 Analytic Continuation Along Paths ..................... 512
3.4 Analytic Continuation and Homotopy ................... 517
3.5 Algebraic Function Elements .............................. 520
3.6 Global Analytic Functions ............................... 527

4 Exercises for Chapter X .................................... 535

Appendix A Background on Fields .......................... 543
1 Fields ....................................................... 543
1.1 The Field Axioms ......................................... 543
1.2 Subfields ................................................ 544
1.3 Isomorphic Fields ........................................ 544

2 Order in Fields ............................................. 545
2.1 Ordered Fields ........................................... 545
2.2 Complete Ordered Fields ................................. 546
2.3 Implications for Real Sequences ....................... 546
Appendix B  Winding Numbers Revisited  

1 Technical Facts About Winding Numbers  
1.1 The Geometric Interpretation  
1.2 Winding Numbers and Jordan Curves  

Index