THE ANALYTICAL AND NUMERICAL SOLUTION OF ELECTRIC AND MAGNETIC FIELDS

K. J. Binns
University of Liverpool, UK

P. J. Lawrenson
Switched Reluctance Drives Ltd
Leeds, UK

C. W. Trowbridge
Vector Fields Ltd,
Kidlington, UK

JOHN WILEY & SONS
Chichester • New York • Brisbane • Toronto • Singapore
CONTENTS

Preface xiii

1 Basic Field Theory 1

1.1 Introduction 1

1.2 Quasi-static electromagnetic field equations 3
  1.2.1 Magnetic vector potential 5
  1.2.2 Magnetic scalar potential 6
  1.2.3 Electric vector potential 7
  1.2.4 \( \mathbf{H} \) field formulations 8

1.3 Static fields 8
  1.3.1 Magnetostatics 8
  1.3.2 Electrostatics 8
  1.3.3 Current flow 9

1.4 Integral equation formulations 9
  1.4.1 Volume magnetization integral equation method 9
  1.4.2 The boundary element method 9

1.5 Interface and boundary conditions 10
  1.5.1 Interface condition 10
  1.5.2 Equivalent pole and charge distributions 12

1.6 Calculation of global quantities 13
  1.6.1 Capacitance 13
  1.6.2 Inductance 14

1.7 Forces 15
  1.7.1 Line sources 15
  1.7.2 Distributed sources 15
  1.7.3 Total force acting on a boundary 16
  1.7.4 Force distribution over a boundary 17

1.8 Field equations in partial differential form 17

1.9 Summary 18

References 19

2 Images 21

2.1 Introduction 21

2.2 Plane boundaries 22
  2.2.1 Single plane boundary 22
  2.2.2 Parallel plane boundaries 25
  2.2.3 Intersecting plane boundaries 27
  2.2.4 Inductance of parallel bus-bars near an iron surface 29

2.3 Circular boundaries 30
3 The Solution of Laplace’s Equation by Separation of the Variables

3.1 Introduction

3.2 Circular boundaries
3.2.1 The solution of Laplace’s equation in circular-cylinder coordinates
3.2.2 Iron cylinder influenced by a current
3.2.3 The screening effect of a permeable cylinder
3.2.4 The force between rotor and stator conductors in a cylindrical electrical machine
3.2.5 Specified distribution of potential or potential gradient on the perimeter of a circular boundary

3.3 Rectangular boundaries
3.3.1 Solution of Laplace’s equation in Cartesian coordinates
3.3.2 The semi-infinite strip and the rectangle
3.3.3 Pole profile in the inductor alternator for a sinusoidal flux distribution

3.4 Conclusions

References

4 Fields with Distributed Currents: Poisson and Diffusion Equations

4.1 Introduction

4.2 Current-carrying conductors in free space
4.2.1 Basic method: vector potential of a line current
4.2.2 The field of a rectangular bus-bar
4.2.3 The force between parallel rectangular bus-bars
4.2.4 Filamentary conductors of arbitrary three-dimensional shape: Biot-Savart law
4.2.5 Boundary effects: use of the image method

4.3 Solution of Poisson’s equation in single and double series
4.3.1 Introduction
4.3.2 Single series solutions: Rogowski method
4.3.3 Double series solutions: Roth’s method
4.3.4 Scope of the methods

4.4 Time-dependent fields: eddy currents
4.4.1 Introduction
4.4.2 Treatment of time variation
4.4.3 Harmonic time variation with spatially fixed fields
4.4.4 Harmonic time variation with travelling fields
4.4.5 General time dependence: transients

4.5 Three-dimensional fields
4.5.1 Introduction
4.5.2 Diffusion equation in rectangular coordinates
4.5.3 Diffusion equation in cylindrical polar coordinates

References

5 Conformal Transformation: Basic Ideas

5.1 Conformal transformation and conjugate functions
5.1.1 Conformal transformation

References
5.1.2 The solution of Laplace's equation
5.1.3 The logarithmic function

5.2 Approaching the solution
5.2.1 Choice of origin
5.2.2 Multiple transformations
5.2.3 Field maps
5.2.4 Scale relationships between planes
5.2.5 Conservation of flux and potential
5.2.6 Field strength

5.3 The bilinear transformation
5.3.1 Mapping properties
5.3.2 The cross-ratio
5.3.3 The magnetic field of currents inside an infinitely permeable tube
5.3.4 Capacitance of and voltage gradient between two cylindrical conductors

5.4 The simple Joukowski transformation
5.4.1 The transformation
5.4.2 Flow round a circular hole
5.4.3 Permeable cylinder influenced by a line current

5.5 Curves expressible parametrically: general series transformations
5.5.1 The method
5.5.2 The field outside a charged, conducting boundary of elliptical shape
5.5.3 General series transformations
5.5.4 Field solutions

References

6 Polygonal Boundaries
6.1 Introduction
6.2 Transformation of the upper half plane into the interior of a polygon
6.2.1 The transformation
6.2.2 Polgons with two vertices
6.2.3 Parallel plate capacitor. Rogowski electrode
6.2.4 The choice of corresponding points
6.2.5 Scale relationship between planes
6.2.6 The field of a current in a slot
6.2.7 Negative vertex angles
6.2.8 The forces between the armature and magnet of a contactor
6.2.9 A simple electrostatic lens

6.3 Transformation of the upper half plane into the region exterior to a polygon
6.3.1 The transformation
6.3.2 The field of a charged, conducting plate

6.4 Transformations from a circular to a polygonal boundary
6.4.1 The transformation equations
6.4.2 The field of a line current and a permeable plate of finite cross-section

6.5 Classification of integrals

References

7 General Considerations
7.1 Introduction
7.2 Field sources
7.2.1 Infinite boundaries
7.2.2 Finite boundaries
7.2.3 Distributed sources
CONTENTS

7.3 Curved boundaries
  7.3.1 Rounded corners
  7.3.2 Curvilinear polygons
7.4 Angles not multiples of π/2
7.5 The use of elliptic functions
  7.5.1 Elliptic integrals and functions
  7.5.2 Two finite charged plates
  7.5.3 Elliptic integrals of the third kind
  7.5.4 The occurrence of elliptic functions
7.6 Numerical methods
  7.6.1 Classes of integral that arise in numerical evaluation
7.7 Non-equipotential boundaries
  7.7.1 Boundary value problems of the first kind
  7.7.2 Boundary value problems of the second and mixed kinds
References

8 Computational Modelling—Basic Methods

8.1 Introduction and engineering objectives
8.2 Mathematical models
  8.2.1 Differential equations
  8.2.2 Integral equations
  8.2.3 Finite difference methods
8.3 The finite element method
  8.3.1 Elements
  8.3.2 Variational method
8.4 Method of weighted residuals
  8.4.1 Collocation methods
  8.4.2 Galerkin method (global)
  8.4.3 Comparison of weighting functions and generalization
8.5 Concluding remarks
References

9 Two-dimensional Static Linear Problems

9.1 Discretizing Poisson's equation by finite elements
  9.1.1 Galerkin method and local basis functions
  9.1.2 Element continuity and element coefficients
9.2 Assembling the matrix
9.3 Boundary conditions
9.4 Linear algebra and solving the system equations
9.5 Summary
9.6 The solution of Poisson equation by boundary elements
9.7 Weighted residual and boundary elements
  9.7.1 Boundary element coefficients and system matrix
  9.7.2 Multiple regions and open boundary problems
9.8 Alternative boundary integral methods
9.9 Axisymmetric form of Poisson's equation
9.10 Scalar potential solution
9.11 Vector potential solution
9.12 Analytic transformations to improve the accuracy near the axis
9.13 Concluding remarks
References
13 Electromagnetic Software Environment

13.1 Introduction 385
13.2 Elements of a CAD system 386
13.3 Pre-processing and mesh generation 387
   13.3.1 Semi-automatic 387
   13.3.2 Manual 388
   13.3.3 Mapping methods 390
   13.3.4 Automatic mesh generation 391
13.4 Solution processor 396
   13.4.1 Field equations 396
   13.4.2 Methods for solving the algebraic system of equations 398
   13.4.3 Gaussian elimination revisited and remarks on conditioning 399
   13.4.4 Iterative methods 402
   13.4.5 A comparison of methods 405
13.5 Error estimation 406
   13.5.1 Error estimators for finite element calculations 407
   13.5.2 Local error estimation using interpolation theory 410
13.6 Open boundary problems 411
13.7 Mesh adaptation solutions 414
13.8 Post-processing and optimization 414
13.9 Summary 418

References 419

Appendix 1 A11 Potential and flux functions 423
   A1.1 Potential and flux functions 423
   A1.2 The magnetic field of line currents 424
   A1.3 Conjugate functions 427
Appendix 2 The Field Inside a Highly Permeable Rectangular Conductor 433
Appendix 3 Table of Transformations 437
   A3.1 Transformations to the upper half-plane 438
   A3.2 Other transformations 451
Appendix 4 Useful Vector Identities and Integral Theorems 453
   A4.1 Vector identities 453
   A4.2 Integral theorems 453
Appendix 5 Quadrature Rules for Numerical Integration 455
Appendix 6 Uniqueness of Scalar Potentials 457

Bibliography 459

Index 465