THE FINITE ELEMENT METHOD IN ELECTROMAGNETICS

JIANMING JIN
Department of Electrical Engineering
and Computer Science
The University of Michigan
Ann Arbor, Michigan

A Wiley-Interscience Publication
JOHN WILEY & SONS, INC.
New York • Chichester • Brisbane • Toronto • Singapore
CONTENTS

Preface xv
Acknowledgments xix

1 A Brief Review of Basic Electromagnetics 1

1.1 Maxwell's Equations, 1
  1.1.1 The General Differential Form, 1
  1.1.2 Electro- and Magnetostatic Fields, 2
  1.1.3 Time-Harmonic Fields, 3
  1.1.4 Constitutive Relations, 3

1.2 Scalar and Vector Potentials, 3
  1.2.1 Scalar Potential for Electrostatic Field, 4
  1.2.2 Vector Potential for Magnetostatic Field, 4

1.3 Wave Equations, 4
  1.3.1 Vector Wave Equations, 5
  1.3.2 Scalar Wave Equations, 5

1.4 Boundary Conditions, 6
  1.4.1 At the Interface between Two Media, 6
  1.4.2 At a Perfectly Conducting Surface, 6
  1.4.3 At an Imperfectly Conducting Surface, 7

1.5 Radiation Conditions, 7
  1.5.1 The Sommerfeld Radiation Condition, 8
2 Introduction to the Finite Element Method

2.1 Classic Methods for Boundary-Value Problems, 11
   2.1.1 Boundary-Value Problems, 11
   2.1.2 The Ritz Method, 12
   2.1.3 Galerkin's Method, 14

2.2 A Simple Example, 15
   2.2.1 Description of the Problem, 15
   2.2.2 Solution via the Ritz Method, 16
   2.2.3 Solution via Galerkin's Method, 18
   2.2.4 Solution Using Subdomain Expansion Functions: The Finite Element Method, 19

2.3 Basic Steps of the Finite Element Method, 23
   2.3.1 Domain Discretization, 23
   2.3.2 Selection of Interpolation Functions, 26
   2.3.3 Formulation of the System of Equations, 26
   2.3.4 Solution of the System of Equations, 29

2.4 An Alternative Presentation of the Finite Element Formulation, 30

References, 32

3 One-Dimensional Finite Element Analysis

3.1 The Boundary-Value Problem, 33

3.2 The Variational Formulation, 34

3.3 Finite Element Analysis, 37
   3.3.1 Discretization and Interpolation, 37
   3.3.2 Formulation via the Ritz Method, 38
   3.3.3 Formulation via Galerkin's Method, 45
   3.3.4 Solution of the System of Equations, 47

3.4 Plane Wave Reflection by a Metal-Backed Dielectric Slab, 48
   3.4.1 Description of Problem, 48
   3.4.2 The Analytical Solution, 50
   3.4.3 Finite Element Solution, 51
   3.4.4 Numerical Results, 53

3.5 Scattering by a Smooth Convex Impedance Cylinder, 55
   3.5.1 Formulation of the OSRC Method, 55
   3.5.2 Finite Element Solution, 58
3.6 Higher-Order Elements, 60
    3.6.1 Quadratic Elements, 60
    3.6.2 Cubic Elements, 66
    3.6.3 Accuracy versus Element Order, 69

References, 71

4 Two-Dimensional Finite Element Analysis 72
    4.1 The Boundary-Value Problem, 72
    4.2 The Variational Formulation, 74
    4.3 Finite Element Analysis, 77
        4.3.1 Domain Discretization, 77
        4.3.2 Elemental Interpolation, 79
        4.3.3 Formulation via the Ritz Method, 81
        4.3.4 Formulation via Galerkin's Method, 89
        4.3.5 A Sample Computer Program, 93
        4.3.6 Solution of the System of Equations, 95
    4.4 Application to Electrostatic Problems, 96
        4.4.1 Two-Dimensional Case, 96
        4.4.2 Axisymmetric Case, 99
    4.5 Application to Magnetostatic Problems, 102
        4.5.1 Two-Dimensional Case, 102
        4.5.2 Axisymmetric Case, 103
    4.6 Application to Time-Harmonic Problems, 105
        4.6.1 Discontinuity in a Parallel-Plate Waveguide, 105
        4.6.2 Scattering Analysis Using Absorbing Boundary Conditions, 107
    4.7 Higher-Order Elements, 123
        4.7.1 Quadratic Triangular Elements, 123
        4.7.2 Construction of Interpolation Functions, 126
        4.7.3 Numerical Integration, 130
        4.7.4 Accuracy versus Element Order, 130

References, 135

5 Three-Dimensional Finite Element Analysis 136
    5.1 The Boundary-Value Problem, 136
    5.2 The Variational Formulation, 137
    5.3 Finite Element Analysis, 138
        5.3.1 Domain Discretization, 138
        5.3.2 Elemental Interpolation, 139
5.3.3 Formulation via the Ritz Method, 141
5.3.4 Formulation via Galerkin's Method, 144

5.4 Rectangular Brick Elements, 146

5.5 Application to Electrostatic Problems, 150

5.6 Application to Magnetostatic Problems, 151
  5.6.1 Problem Description, 151
  5.6.2 The Variational Formulation, 151
  5.6.3 Finite Element Analysis, 152
  5.6.4 On the Uniqueness of Solutions, 155

5.7 Application to Time-Harmonic Field Problems, 158
  5.7.1 Problem Description, 158
  5.7.2 The Variational Formulation, 159
  5.7.3 Treatment of Boundary and Interface Conditions, 161
  5.7.4 The Problem of Spurious Solutions, 164
  5.7.5 The Problem of Field Singularities, 171
  5.7.6 Conclusions, 174

References, 175

6 Variational Principles for Electromagnetics

6.1 Standard Variational Principle, 178

6.2 Modified Variational Principle, 184

6.3 Generalized Variational Principle, 188

6.4 Concluding Remarks, 191

References, 192

7 Eigenvalue Problems: Waveguides and Cavities

7.1 Scalar Formulations for Closed Waveguides, 194
  7.1.1 Homogeneous Waveguides, 194
  7.1.2 Inhomogeneous Waveguides, 197
  7.1.3 Anisotropic Waveguides, 207
  7.1.4 An Approximate Solution, 209

7.2 Vector Formulations for Closed Waveguides, 213
  7.2.1 Formulation in Terms of Three Components, 214
  7.2.2 Formulation in Terms of Transverse Components, 217
  7.2.3 Comments on Vector Formulations, 221

7.3 Open Waveguides, 223
### 7.4 Three-Dimensional Cavities, 226

References, 227

### 8 Vector Finite Elements

8.1 Two-Dimensional Edge Elements, 232
   8.1.1 Rectangular Elements, 232
   8.1.2 Triangular Elements, 234
   8.1.3 Quadrilateral Elements, 237
   8.1.4 Evaluation of Elemental Matrices, 240

8.2 Waveguide Problem Revisited, 244

8.3 Three-Dimensional Edge Elements, 247
   8.3.1 Brick Elements, 247
   8.3.2 Tetrahedral Elements, 251
   8.3.3 Hexahedral Elements, 252
   8.3.4 Evaluation of Elemental Matrices, 254

8.4 Cavity Problem Revisited, 258

8.5 Waveguide Discontinuities, 263

8.6 Scattering Computation Using Vector ABC, 267

8.7 Conclusions, 277

References, 278

### 9 Finite Element-Boundary Integral Methods

9.1 Scattering by Two-Dimensional Cavity-Backed Apertures, 283
   9.1.1 Formulation for $E_z$-Polarization, 283
   9.1.2 Formulation for $H_z$-Polarization, 291
   9.1.3 Numerical Examples, 295

9.2 Scattering by Two-Dimensional Cylindrical Structures, 298
   9.2.1 Boundary Integral Formulation, 299
   9.2.2 Finite Element Formulation, 302
   9.2.3 Numerical Examples, 306

9.3 Scattering by Three-Dimensional Cavity-Backed Apertures, 311
   9.3.1 Boundary Integral Formulation, 311
   9.3.2 Finite Element Formulation, 315
   9.3.3 Numerical Results, 319

9.4 Radiation by Microstrip Patch Antennas in a Cavity, 320
   9.4.1 Problem Formulation, 320
   9.4.2 Modeling of Antenna Feeds and Loads, 324
   9.4.3 Numerical Results, 325
9.5 Scattering by General Three-Dimensional Bodies, 329
   9.5.1 Boundary Integral Formulation, 329
   9.5.2 Finite Element Formulation, 331
   9.5.3 Numerical Results, 334

9.6 Solution of the Finite Element–Boundary Integral System, 335

9.7 Elimination of Interior Resonances, 338

9.8 Alternative Finite Element–Boundary Integral Formulations, 345
   9.8.1 Two-Boundary Formulation, 346
   9.8.2 Formulation Based on the Equivalence Principle, 347

References, 349

10 Finite Elements and Eigenfunction Expansion 353

10.1 Discontinuities in Waveguides, 354
   10.1.1 Discontinuity in a Parallel-Plate Waveguide, 354
   10.1.2 Discontinuity in a Rectangular Waveguide, 357

10.2 Open-Region Scattering, 361
   10.2.1 Two-Dimensional Scattering, 362
   10.2.2 Three-Dimensional Scattering, 363

10.3 Coupled Pair of Basis Functions: The Unimoment Method, 367
   10.3.1 Two-Dimensional Formulation, 368
   10.3.2 Three-Dimensional Formulation, 371

10.4 Finite Element–Extended Boundary Condition Method, 373
   10.4.1 Two-Dimensional Formulation, 373
   10.4.2 Three-Dimensional Formulation, 377

References, 379

11 Solution of Finite Element Equations 381

11.1 Decomposition Methods, 383
   11.1.1 LU Decomposition, 383
   11.1.2 LDL^T Decomposition, 387

11.2 Conjugate Gradient Methods, 396
   11.2.1 Derivation of the Conjugate Gradient Method, 396
   11.2.2 Extension to the Biconjugate Gradient Method, 404
   11.2.3 Matrix–Vector Product Computation, 408

11.3 Solution of Eigenvalue Problems, 410
   11.3.1 Standard Eigenvalue Problems, 410
11.3.2 Generalized Eigenvalue Problems, 415
References, 416

Appendix A Vector Identities and Integral Theorems 418
   A.1 Vector Identities, 418
   A.2 Integral Theorems, 419

Appendix B The Ritz Procedure for Complex-Valued Problems 421

Appendix C Absorbing Boundary Conditions 423
   C.1 Two-Dimensional Absorbing Boundary Conditions, 423
   C.2 Three-Dimensional Absorbing Boundary Conditions, 427
   C.3 Fictitious Absorber: An Alternative Approach, 431
       References, 435

Index 437