Aspects of Multivariate Statistical Theory

ROBB J. MUIRHEAD
Professor of Statistics
University of Michigan
Contents

TABLES xvii

COMMONLY USED NOTATION xix

1. THE MULTIVARIATE NORMAL AND RELATED DISTRIBUTIONS 1

1.1. Introduction, 1
1.2. The Multivariate Normal Distribution, 2
   1.2.1. Definition and Properties, 2
   1.2.2. Asymptotic Distributions of Sample Means and Covariance
          Matrices, 15
1.3. The Noncentral $\chi^2$ and $F$ Distributions, 20
1.4. Some Results on Quadratic Forms, 26
1.5. Spherical and Elliptical Distributions, 32
1.6. Multivariate Cumulants, 40
Problems, 42

2. JACOBIANS, EXTERIOR PRODUCTS, KRONECKER PRODUCTS, AND RELATED TOPICS 50

2.1. Jacobians, Exterior Products, and Related Topics, 50
   2.1.1. Jacobians and Exterior Products, 50
   2.1.2. The Multivariate Gamma Function, 61
   2.1.3. More Jacobians, 63
   2.1.4. Invariant Measures, 67
2.2. Kronecker Products, 73
Problems, 76

xi
3. SAMPLES FROM A MULTIVARIATE NORMAL DISTRIBUTION, AND THE WISHART AND MULTIVARIATE BETA DISTRIBUTIONS

3.1. Samples From a Multivariate Normal Distribution and Maximum Likelihood Estimation of the Parameters, 79

3.2. The Wishart Distribution, 85
   3.2.1. The Wishart Density Function, 85
   3.2.2. Characteristic Function, Moments, and Asymptotic Distribution, 87
   3.2.3. Some Properties of the Wishart Distribution, 91
   3.2.4. Bartlett's Decomposition and the Generalized Variance, 99
   3.2.5. The Latent Roots of a Wishart Matrix, 103

3.3. The Multivariate Beta Distribution, 108
Problems, 112

4. SOME RESULTS CONCERNING DECISION-THEORETIC ESTIMATION OF THE PARAMETERS OF A MULTIVARIATE NORMAL DISTRIBUTION

4.1. Introduction, 121
4.2. Estimation of the Mean, 122
4.3. Estimation of the Covariance Matrix, 128
4.4. Estimation of the Precision Matrix, 136
Problems, 141

5. CORRELATION COEFFICIENTS

5.1. Ordinary Correlation Coefficients, 144
   5.1.1. Introduction, 144
   5.1.2. Joint and Marginal Distributions of Sample Correlation Coefficients in the Case of Independence, 145
   5.1.3. The Non-null Distribution of a Sample Correlation Coefficient in the Case of Normality, 151
   5.1.4. Asymptotic Distribution of a Sample Correlation Coefficient From an Elliptical Distribution, 157
   5.1.5. Testing Hypotheses about Population Correlation Coefficients, 160

5.2. The Multiple Correlation Coefficient, 164
   5.2.1. Introduction, 164
   5.2.2. Distribution of the Sample Multiple Correlation Coefficient in the Case of Independence, 167
5.2.3. The Non-null Distribution of a Sample Multiple Correlation Coefficient in the Case of Normality, 171
5.2.4. Asymptotic Distributions of a Sample Multiple Correlation Coefficient from an Elliptical Distribution, 179
5.2.5. Testing Hypotheses about a Population Multiple Correlation Coefficient, 185
5.3. Partial Correlation Coefficients, 187
Problems, 189

6. INVARIANT TESTS AND SOME APPLICATIONS
6.1. Invariance and Invariant Tests, 196
6.2. The Multiple Correlation Coefficient and Invariance, 206
6.3. Hotelling's $T^2$ Statistic and Invariance, 211
Problems, 219

7. ZONAL POLYNOMIALS AND SOME FUNCTIONS OF MATRIX ARGUMENT
7.1. Introduction, 225
7.2. Zonal Polynomials, 227
7.2.1. Definition and Construction, 227
7.2.2. A Fundamental Property, 239
7.2.3. Some Basic Integrals, 246
7.3. Hypergeometric Functions of Matrix Argument, 258
7.4. Some Results on Special Hypergeometric Functions, 262
7.5. Partial Differential Equations for Hypergeometric Functions, 266
7.6. Generalized Laguerre Polynomials, 281
Problems, 286

8. SOME STANDARD TESTS ON COVARIANCE MATRICES AND MEAN VECTORS
8.1. Introduction, 291
8.2. Testing Equality of $r$ Covariance Matrices, 291
8.2.1. The Likelihood Ratio Statistic and Invariance, 291
8.2.2. Unbiasedness and the Modified Likelihood Ratio Test, 296
8.2.3. Central Moments of the Modified Likelihood Ratio Statistic, 301
8.2.4. The Asymptotic Null Distribution of the Modified Likelihood Ratio Statistic, 303
8.2.5. Noncentral Moments of the Modified Likelihood Ratio Statistic when \( r = 2 \), 311
8.2.6. Asymptotic Non-null Distributions of the Modified Likelihood Ratio Statistic when \( r = 2 \), 316
8.2.7. The Asymptotic Null Distribution of the Modified Likelihood Ratio Statistic for Elliptical Samples, 329
8.2.8. Other Test Statistics, 331
8.3. The Sphericity Test, 333
  8.3.1. The Likelihood Ratio Statistic; Invariance and Unbiasedness, 333
  8.3.2. Moments of the Likelihood Ratio Statistic, 339
  8.3.3. The Asymptotic Null Distribution of the Likelihood Ratio Statistic, 343
  8.3.4. Asymptotic Non-null Distributions of the Likelihood Ratio Statistic, 344
  8.3.5. The Asymptotic Null Distribution of the Likelihood Ratio Statistic for an Elliptical Sample, 351
  8.3.6. Other Test Statistics, 353
8.4. Testing That a Covariance Matrix Equals a Specified Matrix, 353
  8.4.1. The Likelihood Ratio Test and Invariance, 353
  8.4.2. Unbiasedness and the Modified Likelihood Ratio Test, 356
  8.4.3. Moments of the Modified Likelihood Ratio Statistic, 358
  8.4.4. The Asymptotic Null Distribution of the Modified Likelihood Ratio Statistic, 359
  8.4.5. Asymptotic Non-null Distributions of the Modified Likelihood Ratio Statistic, 362
  8.4.6. The Asymptotic Null Distribution of the Modified Likelihood Ratio Statistic for an Elliptical Sample, 364
  8.4.7. Other Test Statistics, 365
8.5. Testing Specified Values for the Mean Vector and Covariance Matrix, 366
  8.5.1. The Likelihood Ratio Test, 366
  8.5.2. Moments of the Likelihood Ratio Statistic, 369
  8.5.3. The Asymptotic Null Distribution of the Likelihood Ratio Statistic, 370
  8.5.4. Asymptotic Non-null Distributions of the Likelihood Ratio Statistic, 373
Problems, 376

9. PRINCIPAL COMPONENTS AND RELATED TOPICS 380
  9.1. Introduction, 380
  9.2. Population Principal Components, 381
  9.3. Sample Principal Components, 384
9.4. The Joint Distribution of the Latent Roots of a Sample Covariance Matrix, 388
9.5. Asymptotic Distributions of the Latent Roots of a Sample Covariance Matrix, 390
9.6. Some Inference Problems in Principal Components, 405
9.7. Distributions of the Extreme Latent Roots of a Sample Covariance Matrix, 420
Problems, 426

10. THE MULTIVARIATE LINEAR MODEL

10.1. Introduction, 429
10.3. The Noncentral Wishart Distribution, 441
10.4. Joint Distributions of Latent Roots in MANOVA, 449
10.5. Distributional Results for the Likelihood Ratio Statistic, 455
  10.5.1. Moments, 455
  10.5.2. Null Distribution, 457
  10.5.3. The Asymptotic Null Distribution, 458
  10.5.4. Asymptotic Non-null Distributions, 460
10.6 Other Test Statistics, 465
  10.6.1. Introduction, 465
  10.6.2. The $T^2_0$ Statistic, 466
  10.6.3. The $V$ Statistic, 479
  10.6.4. The Largest Root, 481
  10.6.5. Power Comparisons, 484
10.7. The Single Classification Model, 485
  10.7.1. Introduction, 485
  10.7.2. Multiple Discriminant Analysis, 488
  10.7.3. Asymptotic Distributions of Latent Roots in MANOVA, 492
  10.7.4. Determining the Number of Useful Discriminant Functions, 499
  10.7.5. Discrimination Between Two Groups, 504
10.8. Testing Equality of $p$ Normal Populations, 507
  10.8.1. The Likelihood Ratio Statistic and Moments, 507
  10.8.2. The Asymptotic Null Distribution of the Likelihood Ratio Statistic, 512
  10.8.3. An Asymptotic Non-null Distribution of the Likelihood Ratio Statistic, 513
Problems, 517
11. TESTING INDEPENDENCE BETWEEN \( k \) SETS OF VARIABLES AND CANONICAL CORRELATION ANALYSIS

11.1. Introduction, 526

11.2. Testing Independence of \( k \) Sets of Variables, 526
   11.2.1. The Likelihood Ratio Statistic and Invariance, 526
   11.2.2. Central Moments of the Likelihood Ratio Statistic, 532
   11.2.3. The Null Distribution of the Likelihood Ratio Statistic, 533
   11.2.4. The Asymptotic Null Distribution of the Likelihood Ratio Statistic, 534
   11.2.5. Noncentral Moments of the Likelihood Ratio Statistic when \( k = 2 \), 536
   11.2.6. Asymptotic Non-null Distributions of the Likelihood Ratio Statistic when \( k = 2 \), 542
   11.2.7. The Asymptotic Null Distribution of the Likelihood Ratio Statistic for Elliptical Samples, 546
   11.2.8. Other Test Statistics, 548

11.3. Canonical Correlation Analysis, 548
   11.3.1. Introduction, 548
   11.3.2. Population Canonical Correlation Coefficients and Canonical Variables, 549
   11.3.3. Sample Canonical Correlation Coefficients and Canonical Variables, 555
   11.3.4. Distributions of the Sample Canonical Correlation Coefficients, 557
   11.3.5. Asymptotic Distributions of the Sample Canonical Correlation Coefficients, 562
   11.3.6. Determining the Number of Useful Canonical Variables, 567

Problems, 569

APPENDIX. SOME MATRIX THEORY

   A1. Introduction, 572
   A2. Definitions, 572
   A3. Determinants, 575
   A4. Minors and Cofactors, 579
   A5. Inverse of a Matrix, 579
   A6. Rank of a Matrix, 582
   A7. Latent Roots and Latent Vectors, 582
   A8. Positive Definite Matrices, 585
   A9. Some Matrix Factorizations, 586

BIBLIOGRAPHY

INDEX