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Integral Transforms in Geophysics

With 71 Figures

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# Contents

## Part I  Cauchy-Type Integrals in the Theory of a Plane Geopotential Field

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