V.E. Zakharov (Ed.)

What Is Integrability?

With Contributions by
F. Calogero  N. Ercolani  H. Flaschka
V.A. Marchenko  A.V. Mikhailov
A.C. Newell  E.I. Schulman  A.B. Shabat
E.D. Siggia  V.V. Sokolov  M. Tabor
A.P. Veselov  V.E. Zakharov

Springer-Verlag
Berlin Heidelberg New York London
Paris Tokyo Hong Kong Barcelona
# Contents

Why Are Certain Nonlinear PDEs Both Widely Applicable and Integrable?  
*By F. Calogero*  

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Summary</td>
<td>1</td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1. The Main Ideas in an Illustrative Context</td>
<td>4</td>
</tr>
<tr>
<td>2. Survey of Model Equations</td>
<td>19</td>
</tr>
<tr>
<td>3. C-Integrable Equations</td>
<td>33</td>
</tr>
<tr>
<td>4. Envoi</td>
<td>56</td>
</tr>
<tr>
<td>Addendum</td>
<td>57</td>
</tr>
<tr>
<td>References</td>
<td>61</td>
</tr>
</tbody>
</table>

Painlevé Property and Integrability  
*By N. Ercolani and E.D. Siggia*  

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Background</td>
<td>63</td>
</tr>
<tr>
<td>1.1 Motivation</td>
<td>63</td>
</tr>
<tr>
<td>1.2 History</td>
<td>64</td>
</tr>
<tr>
<td>2. Integrability</td>
<td>64</td>
</tr>
<tr>
<td>3. Riccati Example</td>
<td>65</td>
</tr>
<tr>
<td>4. Balances</td>
<td>66</td>
</tr>
<tr>
<td>5. Elliptic Example</td>
<td>67</td>
</tr>
<tr>
<td>6. Augmented Manifold</td>
<td>68</td>
</tr>
<tr>
<td>7. Argument for Integrability</td>
<td>69</td>
</tr>
<tr>
<td>8. Separability</td>
<td>70</td>
</tr>
<tr>
<td>References</td>
<td>72</td>
</tr>
</tbody>
</table>

Integrability  
*By H. Flaschka, A.C. Newell and M. Tabor*  

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Integrability</td>
<td>73</td>
</tr>
<tr>
<td>2. Introduction to the Method</td>
<td>80</td>
</tr>
<tr>
<td>2.1 The WTC Method for Partial Differential Equations</td>
<td>81</td>
</tr>
<tr>
<td>2.2 The WTC Method for Ordinary Differential Equations</td>
<td>84</td>
</tr>
<tr>
<td>2.3 The Nature of $\phi$</td>
<td>86</td>
</tr>
<tr>
<td>2.4 Truncated Versus Non-truncated Expansions</td>
<td>89</td>
</tr>
<tr>
<td>3. The Integrable Hénon-Heiles System: A New Result</td>
<td>90</td>
</tr>
<tr>
<td>3.1 The Lax Pair</td>
<td>90</td>
</tr>
</tbody>
</table>
Contents

3.2 The Algebraic Curve and Integration of the Equations of Motion ......................................... 92
3.3 The Role of the Rational Solutions in the Painlevé Expansions ............................................. 95
4. A Mikhailov and Shabat Example ................................................................................................. 97
5. Some Comments on the KdV Hierarchy ...................................................................................... 98
6. Connection with Symmetries and Algebraic Structure ................................................................. 99
7. Integrating the Nonintegrable ..................................................................................................... 106
References ...................................................................................................................................... 113

The Symmetry Approach to Classification of Integrable Equations .................................................... 115
By A.V. Mikhailov, A.B. Shabat and V.V. Sokolov

Introduction ....................................................................................................................................... 115
1. Basic Definitions and Notations ................................................................................................. 116
  1.1 Classical and Higher Symmetries ......................................................................................... 116
  1.2 Local Conservation Laws .................................................................................................. 121
  1.3 PDEs and Infinite-Dimensional Dynamical Systems ......................................................... 123
  1.4 Transformations ................................................................................................................ 124
2. The Burgers Type Equations ........................................................................................................ 129
  2.1 Classification in the Scalar Case ......................................................................................... 129
  2.2 Systems of Burgers Type Equations .................................................................................. 135
  2.3 Lie Symmetries and Differential Substitutions ................................................................... 142
3. Canonical Conservation Laws ....................................................................................................... 146
  3.1 Formal Symmetries ............................................................................................................. 146
  3.2 The Case of a Vector Equation ............................................................................................ 152
  3.3 Integrability Conditions ....................................................................................................... 158
4. Integrable Equations .................................................................................................................... 161
  4.1 Scalar Third Order Equations ............................................................................................. 161
  4.2 Scalar Fifth Order Equations .............................................................................................. 170
  4.3 Schrödinger Type Equations ............................................................................................... 173
Historical Remarks .......................................................................................................................... 182
References .......................................................................................................................................... 183

Integrability of Nonlinear Systems and Perturbation Theory ......................................................... 185
By V.E. Zakharov and E.I. Schulman

1. Introduction .................................................................................................................................. 185
2. General Theory ............................................................................................................................ 187
  2.1 The Formal Classical Scattering Matrix in the Soliton-less Sector of Rapidly Decreasing Initial Conditions ................................................................. 187
  2.2 Infinite-Dimensional Generalization of Poincaré’s Theorem. Definition of Degenerative Dispersion Laws ......................................................................................... 193
  2.3 Properties of Degenerative Dispersion Laws ....................................................................... 197
## Contents

2.4 Properties of Singular Elements of a Classical Scattering Matrix. Properties of Asymptotic States .......................... 205

2.5 The Integrals of Motion .................................................. 209

2.6 The Integrability Problem in the Periodic Case. Action-Angle Variables ..................................................... 213

3. Applications to Particular Systems ......................................... 222

3.1 The Derivation of Universal Models ....................................... 222

3.2 Kadomtsev-Petviashvili and Veselov-Novikov Equations .............. 227

3.3 Davey-Stewartson-Type Equations. The Universality of the Davey-Stewartson Equation in the Scope of Solvable Models ............ 230

3.4 Applications to One-Dimensional Equations .............................. 232

Appendix I ................................................................. 236

Proofs of the Local Theorems (of Uniqueness and Others from Sect.2.3) .......................................................... 236

Appendix II ................................................................. 244

Proof of the Global Theorem for Degenerative Dispersion Laws ............ 244

Conclusion ................................................................. 247

References ................................................................. 249

### What Is an Integrable Mapping?

*By A.P. Veselov*

Introduction ............................................................. 251

1. Integrable Polynomial and Rational Mappings ......................... 252

1.1 Polynomial Mapping of C: What Is Its Integrability? ................ 252

1.2 Commuting Polynomial Mappings of $C^N$ and Simple Lie Algebras .... 254

1.3 Commuting Rational Mappings of $C P^n$ ............................... 257

1.4 Commuting Cremona Mappings of $C^2$ .................................. 258

1.5 Euler-Chasles Correspondences and the Yang-Baxter Equation .......... 260

2. Integrable Lagrangean Mappings with Discrete Time .................. 261

2.1 Hamiltonian Theory .................................................... 261

2.2 Heisenberg Chain with Classical Spins and the Discrete Analog of the C. Neumann System ................................. 263

2.3 The Billiard in Quadrics ................................................. 264

2.4 The Discrete Analog of the Dynamics of the Top ...................... 266

2.5 Connection with the Spectral Theory of the Difference Operators: A Discrete Analogue of the Moser-Trubowitz Isomorphism .......... 267

Appendix A ................................................................. 269

Appendix B ................................................................. 270

References ................................................................. 270
The Cauchy Problem for the KdV Equation
with Non-Decreasing Initial Data .................................................. 273
By V.A. Marchenko
1. Reflectionless Potentials .......................................................... 274
2. Closure of the Sets $B(-\mu^2)$ .................................................... 291
3. The Inverse Problem ............................................................... 301
References .................................................................................... 318

Subject Index .................................................................................. 319