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Problems and Theorems in Linear Algebra
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7. Bases of a vector space. Linear independence

7.2. **Theorem.** Let \(x_1, \ldots, x_n\) and \(y_1, \ldots, y_n\) be two bases, \(1 \leq k \leq n\). Then \(k\) of the vectors \(y_1, \ldots, y_n\) can be interchanged with some \(k\) of the vectors \(x_1, \ldots, x_n\) so that we get again two bases.

7.3. **Theorem.** Let \(T : V \rightarrow V\) be a linear operator such that the vectors \(\xi, T\xi, \ldots, T^n\xi\) are linearly dependent for every \(\xi \in V\). Then the operators \(I, T, \ldots, T^n\) are linearly dependent.

**Problems**

8. The rank of a matrix

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8.3. **Theorem.** Let \(U\) be a linear subspace of the space \(M_{n,m}\) of \(n \times m\) matrices, and \(r \leq m \leq n\).

If \(\text{rank } X < r\) for any \(X \in U\) then \(\dim \ U < nr\).

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**Problems**

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9.6.1. **Theorem.** A set of \(k\)-dimensional subspaces of \(V\) is such that any two of these subspaces have a common \((k-1)\)-dimensional subspace. Then either all these subspaces have a common \((k-1)\)-dimensional subspace or all of them are contained in the same \((k+1)\)-dimensional subspace.

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b) The operators $A_n$ and $A_s$ are unique; besides, $A_s = S(A)$ and $A_n = N(A)$ for some polynomials $S$ and $N$.

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Problems

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21.2. Theorem. Any skew-symmetric bilinear form can be expressed as

$$
\sum_{k=1}^{r} (x_{2k-1}y_{2k} - x_{2k}y_{2k-1}).
$$

Problems

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Problems

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23.2. Theorem. If an operator $A$ is normal then there exists a polynomial $P$ such that $A^* = P(A)$.

Problems

24. Nilpotent matrices

24.2.1. Theorem. Let $A$ be an $n \times n$ matrix. The matrix $A$ is nilpotent if and only if $\text{tr}(A^p) = 0$ for each $p = 1, \ldots, n$.

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Problems

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25.2.1&2. Theorem. An idempotent operator $P$ is an Hermitian one if and only if a) $\text{Ker} \ P \perp \text{Im} \ P$; or b) $|Px| \leq |x|$ for every $x$.

25.2.3. Theorem. Let $P_1, \ldots, P_n$ be Hermitian, idempotent operators. The operator $P = P_1 + \cdots + P_n$ is an idempotent one if and only if $P_iP_j = 0$ whenever $i \neq j$.

25.4.1. Theorem. Let $V_1 \oplus \cdots \oplus V_k, P_i : V \rightarrow V_i$ be Hermitian idempotent operators, $A = P_1 + \cdots + P_k$. Then $0 < \det A \leq 1$ and $\det A = 1$ if and only if $V_i \perp V_j$ whenever $i \neq j$.

Problems

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Chapter V. Multilinear algebra

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**Problems**

28. Symmetric and skew-symmetric tensors


**28.5.4. Theorem.** Let \( \Lambda_B(t) = 1 + \sum_{q=1}^{n} \text{tr}(A_B^q)t^q \) and \( S_B(t) = 1 + \sum_{q=1}^{n} \text{tr}(S_B^q)t^q \). Then \( S_B(t) = (\Lambda_B(-1))^{-1} \).

**Problems**

29. The Pfaffian

The Pfaffian of principal submatrices of the matrix \( M = ||m_{ij}||_{1}^{2n} \), where \( m_{ij} = (-1)^{i+j+1} \).

**29.2.2. Theorem.** Given a skew-symmetric matrix \( A \) we have

\[
\text{Pf}(A + \lambda^2 M) = \sum_{k=0}^{n} \lambda^{2k} p_k, \quad \text{where} \quad p_k = \sum_{\sigma} A \left( \begin{array}{ccc} \sigma_1 & \cdots & \sigma_{2(n-k)} \\ \sigma_1 & \cdots & \sigma_{2(n-k)} \end{array} \right).
\]

**Problems**

30. Decomposable skew-symmetric and symmetric tensors

**30.2.1. Theorem.** \( x_1 \wedge \cdots \wedge x_k = y_1 \wedge \cdots \wedge y_k \neq 0 \) if and only if \( \text{Span}(x_1, \ldots, x_k) = \text{Span}(y_1, \ldots, y_k) \).

**30.2.2. Theorem.** \( S(x_1 \otimes \cdots \otimes x_k) = S(y_1 \otimes \cdots \otimes y_k) \neq 0 \) if and only if \( \text{Span}(x_1, \ldots, x_k) = \text{Span}(y_1, \ldots, y_k) \).

Plucker relations.

**Problems**

31. The tensor rank

Strassen’s algorithm. The set of all tensors of rank \( \leq 2 \) is not closed. The rank over \( \mathbb{R} \) is not equal, generally, to the rank over \( \mathbb{C} \).

**Problems**

32. Linear transformations of tensor products

A complete description of the following types of transformations of

\[ V^m \otimes (V^*)^n \cong M_{m,n} : \]

1) rank-preserving;
2) determinant-preserving;
3) eigenvalue-preserving;
4) invertibility-preserving.

**Problems**

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**Chapter VI. Matrix inequalities**

33. Inequalities for symmetric and Hermitian matrices

**33.1.1. Theorem.** If \( A > B > 0 \) then \( A^{-1} < B^{-1} \).

**33.1.3. Theorem.** If \( A > 0 \) is a real matrix then

\[
(A^{-1}x, x) = \max_{y}(2(x, y) - (Ay, y)).
\]
33.2.1. **Theorem.** Suppose $A = \begin{pmatrix} A_1 & B \\ B^* & A_2 \end{pmatrix} > 0$. Then $|A| \leq |A_1| \cdot |A_2|$. 
Hadamard's inequality and Szasz's inequality.

33.3.1. **Theorem.** Suppose $\alpha_i > 0$, $\sum_{i=1}^{n} \alpha_i = 1$ and $A_i > 0$. Then

$$|\alpha_1 A_1 + \cdots + \alpha_k A_k| \geq |A_1|^{\alpha_1} + \cdots + |A_k|^{\alpha_k}.$$ 

33.3.2. **Theorem.** Suppose $A_i > 0$, $\alpha_i \in \mathbb{C}$. Then

$$|\det(\alpha_1 A_1 + \cdots + \alpha_k A_k)| \leq \det(|\alpha_1 A_1 + \cdots + \alpha_k A_k|).$$

**Problems**

34. **Inequalities for eigenvalues**

Schur's inequality, Weyl's inequality (for eigenvalues of $A + B$).

34.2.2. **Theorem.** Let $A = \begin{pmatrix} B & C \\ C^* & B \end{pmatrix} > 0$ be an Hermitian matrix, $\alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \cdots \leq \beta_m$ the eigenvalues of $A$ and $B$, respectively. Then $\alpha_i \leq \beta_i \leq \alpha_{n+1-m}$.

34.3. **Theorem.** Let $A$ and $B$ be Hermitian idempotents, $\lambda$ any eigenvalue of $AB$. Then $0 \leq \lambda \leq 1$.

34.4.1. **Theorem.** Let the $\lambda_i$ and $\mu_i$ be the eigenvalues of $A$ and $AA^*$, respectively; let $\sigma_i = \sqrt{\mu_i}$. Let $|\lambda_1| \leq \cdots \leq |\lambda_n|$, where $n$ is the order of $A$. Then $|\lambda_1 \cdots \lambda_m| \leq \sigma_1 \cdots \sigma_m$.

34.4.2. **Theorem.** Let $\sigma_1 \geq \cdots \geq \sigma_n$ and $\tau_1 \geq \cdots \geq \tau_n$ be the singular values of $A$ and $B$. Then $|\text{tr}(AB)| \leq \sum \sigma_i \tau_i$.

**Problems**

35. **Inequalities for matrix norms**

The spectral norm $\|A\|_p$ and the Euclidean norm $\|A\|_e$, the spectral radius $\rho(A)$.

35.1.2. **Theorem.** If a matrix $A$ is normal then $\rho(A) = \|A\|_2$.

35.2. **Theorem.** $\|A\|_e \leq \|A\|_e \leq \sqrt{n} \|A\|_p$.

The invariance of the matrix norm and singular values.

35.3.1. **Theorem.** Let $S$ be an Hermitian matrix. Then $\|A - \frac{A + A^*}{2}\|_e$ does not exceed $\|A - S\|_e$, where $\|\|$ is the Euclidean or operator norm.

35.3.2. **Theorem.** Let $A = US$ be the polar decomposition of $A$ and $W$ a unitary matrix. Then $\|A - U\|_e \leq \|A - W\|_e$ and if $|A| \neq 0$, then the equality is only attained for $W = U$.

**Problems**

36. **Schur’s complement and Hadamard’s product. Theorems of Emily Haynsworth**

36.1.1. **Theorem.** If $A > 0$ then $(A|A_{11}) > 0$.

36.1.4. **Theorem.** If $A_k$ and $B_k$ are the $k$th principal submatrices of positive definite order $m$ matrices $A$ and $B$, then

$$|A + B| \geq |A| \left(1 + \sum_{k=1}^{n-1} \frac{|B_k|}{|A_k|}\right) + |B| \left(1 + \sum_{k=1}^{n-1} \frac{|A_k|}{|B_k|}\right).$$

Hadamard's product $A \circ B$.

36.2.1. **Theorem.** If $A > 0$ and $B > 0$ then $A \circ B > 0$.

Oppenheim's inequality.

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37. **Nonnegative matrices**

Wielandt's theorem

**Problems**
38. Doubly stochastic matrices

Birkhoff's theorem. H. Weyl's inequality.

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Chapter VII. Matrices in algebra and calculus

39. Commuting matrices

The space of solutions of the equation $AX = XA$ with the given $A$ of order $n$.

39.2. Theorem. Any set of commuting diagonalizable operators has a common eigenbasis.

39.3. Theorem. Let $A, B$ be matrices such that $AX = XA$ implies $BX = XB$. Then $B = g(A)$, where $g$ is a polynomial.

Problems

40. Commutators

40.2. Theorem. If $tr A = 0$ then there exist matrices $X$ and $Y$ such that $[X, Y] = A$ and either (1) $tr Y = 0$ and an Hermitian matrix $X$ or (2) $X$ and $Y$ have prescribed eigenvalues.

40.3. Theorem. Let $A, B$ be matrices such that $ad^s_X A = 0$ implies $ad^s_Y B = 0$ for some $s > 0$. Then $B = g(A)$ for a polynomial $g$.

40.4. Theorem. Matrices $A_1, \ldots, A_n$ can be simultaneously triangularized over $C$ if and only if the matrix $p(A_1, \ldots, A_n)[A_1, A_2]$ is a nilpotent one for any polynomial $p(x_1, \ldots, x_n)$ in noncommuting indeterminates.

40.5. Theorem. If $r n k A, B \leq 1$, then $A$ and $B$ can be simultaneously triangularized over $C$.

Problems

41. Quaternions and Cayley numbers. Clifford algebras

Isomorphisms $so(3, R) \cong su(2)$ and $so(4, R) \cong so(3, R) \oplus so(3, R)$. The vector products in $K^3$ and $K^7$. Hurwitz-Radon families of matrices. Hurwitz-Radon number $\rho(2^{t+2d}(2a + 1)) = 2^t + 8d$.

41.7. Theorem. The identity of the form

$$\left( x_1^2 + \cdots + x_m^2 \right) \left( y_1^2 + \cdots + y_m^2 \right) = \left( z_1^2 + \cdots + z_m^2 \right),$$

where $z_i(x, y)$ is a bilinear function, holds if and only if $m \leq \rho(n)$.

41.7.5. Theorem. In the space of real $n \times n$ matrices, a subspace of invertible matrices of dimension $m$ exists if and only if $m \leq \rho(n)$.

Other applications: algebras with norm, vector product, linear vector fields on spheres. Clifford algebras and Clifford modules.

Problems

42. Representations of matrix algebras

Complete reducibility of finite-dimensional representations of $Mat(V^n)$.

Problems

43. The resultant

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Problems

44. The general inverse matrix. Matrix equations

44.3. Theorem. a) The equation $AX = XA = C$ is solvable if and only if the matrices

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix}$$

and

$$\begin{pmatrix} A & C \\ O & B \end{pmatrix}$$

are similar.

b) The equation $AX = YA = C$ is solvable if and only if $rank\begin{pmatrix} A & O \\ O & B \end{pmatrix} = rank\begin{pmatrix} A & C \\ O & B \end{pmatrix}$.

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45. Hankel matrices and rational functions
46. Functions of matrices. Differentiation of matrices

        Differential equation $\dot{X} = AX$ and the Jacobi formula for det $A$.

Problems

47. Lax pairs and integrable systems

48. Matrices with prescribed eigenvalues

    48.1.2. Theorem. For any polynomial $f(x) = x^n + c_1 x^{n-1} + \ldots + c_n$ and any matrix $B$ of order $n-1$ whose characteristic and minimal polynomials coincide there exists a matrix $A$ such that $B$ is a submatrix of $A$ and the characteristic polynomial of $A$ is equal to $f$.

    48.2. Theorem. Given all offdiagonal elements in a complex matrix $A$ it is possible to select diagonal elements $x_1, \ldots, x_n$ so that the eigenvalues of $A$ are given complex numbers; there are finitely many sets $\{x_1, \ldots, x_n\}$ satisfying this condition.

Solutions

Appendix

Eisenstein's criterion, Hilbert's Nullstellensatz.

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