CONTENTS

Part A—Formulation of Variational Principles in Elasticity and Plasticity

INTRODUCTION 3

CHAPTER 1. SMALL DISPLACEMENT THEORY OF ELASTICITY IN RECTANGULAR CARTESIAN COORDINATES 11
1.1. Presentation of a Problem in Small Displacement Theory 11
1.2. Conditions of Compatibility 14
1.3. Stress Functions 16
1.4. Principle of Virtual Work 18
1.5. Approximate Method of Solution Based on the Principle of Virtual Work 19
1.6. Principle of Complementary Virtual Work 22
1.7. Approximate Method of Solution Based on the Principle of Complementary Virtual Work 24
1.8. Relations between Conditions of Compatibility and Stress Functions 28
1.9. Some Remarks 30
Exercises 33

CHAPTER 2. VARIATIONAL PRINCIPLES IN THE SMALL DISPLACEMENT THEORY OF ELASTICITY 42
2.1. Principle of Minimum Potential Energy 42
2.2. Principle of Minimum Complementary Energy 45
2.3. Generalization of the Principle of Minimum Potential Energy 48
2.4. Derived Variational Principles 50
2.5. Rayleigh–Ritz Method—(1) 55
2.6. Variation of the Boundary Conditions and Castigliano’s Theorem 57
2.7. Free Vibrations of an Elastic Body 61
2.8. Rayleigh–Ritz Method—(2) 64
2.9. Some Remarks 67
Exercises 70

CHAPTER 3. FINITE DISPLACEMENT THEORY OF ELASTICITY IN RECTANGULAR CARTESIAN COORDINATES 79
3.1. Analysis of Strain 79
3.2. Analysis of Stress and Equations of Equilibrium 83
3.3. Transformation of the Stress Tensor 86
3.4. Stress–Strain Relations 87
3.5. Presentation of a Problem 88
3.6. Principle of Virtual Work 91
3.7. Strain Energy Function 93
3.8. Principle of Stationary Potential Energy 96
3.9. Generalization of the Principle of Stationary Potential Energy 97
3.11. The Euler Method for Stability Problem 101
3.12. Some Remarks 104
Exercises 105
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Theory of Elasticity in Curvilinear Coordinates</td>
<td>107</td>
</tr>
<tr>
<td>4.1</td>
<td>Geometry before Deformation</td>
<td>107</td>
</tr>
<tr>
<td>4.2</td>
<td>Analysis of Strain and Conditions of Compatibility</td>
<td>112</td>
</tr>
<tr>
<td>4.3</td>
<td>Analysis of Stress and Equations of Equilibrium</td>
<td>115</td>
</tr>
<tr>
<td>4.4</td>
<td>Transformation of the Strain and Stress Tensors</td>
<td>117</td>
</tr>
<tr>
<td>4.5</td>
<td>Stress–Strain Relations in Curvilinear Coordinates</td>
<td>120</td>
</tr>
<tr>
<td>4.6</td>
<td>Principle of Virtual Work</td>
<td>121</td>
</tr>
<tr>
<td>4.7</td>
<td>Principle of Stationary Potential Energy and its Generalizations</td>
<td>123</td>
</tr>
<tr>
<td>4.8</td>
<td>Some Specializations to Small Displacement Theory in Orthogonal Curvilinear Coordinates</td>
<td>125</td>
</tr>
<tr>
<td></td>
<td>Exercises</td>
<td>127</td>
</tr>
<tr>
<td>5</td>
<td>Extensions of the Principle of Virtual Work and Related Variational Principles</td>
<td>138</td>
</tr>
<tr>
<td>5.1</td>
<td>Initial Stress Problems</td>
<td>138</td>
</tr>
<tr>
<td>5.2</td>
<td>Stability Problems of a Body with Initial Stresses</td>
<td>142</td>
</tr>
<tr>
<td>5.3</td>
<td>Initial Strain Problems</td>
<td>144</td>
</tr>
<tr>
<td>5.4</td>
<td>Thermal Stress Problems</td>
<td>147</td>
</tr>
<tr>
<td>5.5</td>
<td>Dynamical Problems</td>
<td>150</td>
</tr>
<tr>
<td>5.6</td>
<td>Dynamical Problems of an Elastic Body with Initial Stresses</td>
<td>156</td>
</tr>
<tr>
<td>5.7</td>
<td>Dynamical Problems of an Unrestrained Elastic Body</td>
<td>161</td>
</tr>
<tr>
<td></td>
<td>Exercises</td>
<td>168</td>
</tr>
<tr>
<td>6</td>
<td>Torsion of Bars</td>
<td>173</td>
</tr>
<tr>
<td>6.1</td>
<td>Saint-Venant Theory of Torsion</td>
<td>173</td>
</tr>
<tr>
<td>6.2</td>
<td>The Principle of Minimum Potential Energy and its Transformation</td>
<td>177</td>
</tr>
<tr>
<td>6.3</td>
<td>Torsion of a Bar with a Hole</td>
<td>180</td>
</tr>
<tr>
<td>6.4</td>
<td>Torsion of a Bar with Initial Stresses</td>
<td>183</td>
</tr>
<tr>
<td>6.5</td>
<td>Upper and Lower Bounds of Torsional Rigidity</td>
<td>188</td>
</tr>
<tr>
<td></td>
<td>Exercises</td>
<td>195</td>
</tr>
<tr>
<td>7</td>
<td>Beams</td>
<td>207</td>
</tr>
<tr>
<td>7.1</td>
<td>Elementary Theory of a Beam</td>
<td>207</td>
</tr>
<tr>
<td>7.2</td>
<td>Bending of a Beam</td>
<td>210</td>
</tr>
<tr>
<td>7.3</td>
<td>Principle of Minimum Potential Energy and its Transformation</td>
<td>214</td>
</tr>
<tr>
<td>7.4</td>
<td>Free Lateral Vibration of a Beam</td>
<td>216</td>
</tr>
<tr>
<td>7.5</td>
<td>Large Deflection of a Beam</td>
<td>220</td>
</tr>
<tr>
<td>7.6</td>
<td>Buckling of a Beam</td>
<td>222</td>
</tr>
<tr>
<td>7.7</td>
<td>Free Lateral Vibration of a Rotating Beam</td>
<td>226</td>
</tr>
<tr>
<td>7.8</td>
<td>A Beam Theory Including the Effect of Transverse Shear Deformation</td>
<td>229</td>
</tr>
<tr>
<td>7.9</td>
<td>Beam with Small Initial Curvature</td>
<td>233</td>
</tr>
<tr>
<td>7.10</td>
<td>Some Remarks</td>
<td>237</td>
</tr>
<tr>
<td></td>
<td>Exercises</td>
<td>239</td>
</tr>
<tr>
<td>8</td>
<td>Plates</td>
<td>253</td>
</tr>
<tr>
<td>8.1</td>
<td>Stretching and Bending of a Plate</td>
<td>253</td>
</tr>
<tr>
<td>8.2</td>
<td>A Problem of Stretching and Bending of a Plate</td>
<td>256</td>
</tr>
<tr>
<td>8.3</td>
<td>Principle of Minimum Potential Energy and its Transformation for the Stretching of a Plate</td>
<td>262</td>
</tr>
<tr>
<td>8.4</td>
<td>Principle of Minimum Potential Energy and its Transformation for the Bending of a Plate</td>
<td>265</td>
</tr>
<tr>
<td>8.5</td>
<td>Large Deflection of a Plate in Stretching and Bending</td>
<td>266</td>
</tr>
<tr>
<td>8.6</td>
<td>Buckling of a Plate</td>
<td>270</td>
</tr>
<tr>
<td>8.7</td>
<td>Thermal Stresses in a Plate</td>
<td>273</td>
</tr>
<tr>
<td>8.8</td>
<td>A Thin Plate Theory Including the Effect of Transverse Shear Deformation</td>
<td>276</td>
</tr>
</tbody>
</table>
CONTENTS

8.9. Thin Shallow Shell 280
8.10. Some Remarks 286
  Exercises 289

CHAPTER 9. SHELLS  
9.1. Geometry before Deformation 306
9.2. Analysis of Strain 311
9.3. Analysis of Strain under the Kirchhoff–Love Hypothesis 315
9.4. A Linearized Thin Shell Theory under the Kirchhoff–Love Hypothesis 317
9.5. Simplified Formulations 322
9.6. A Simplified Linear Theory under the Kirchhoff–Love Hypothesis 324
9.7. A Nonlinear Thin Shell Theory under the Kirchhoff–Love Hypothesis 325
9.8. A Linearized Thin Shell Theory Including the Effect of Transverse Shear Deformations 327
9.9. Some Remarks 330
  Exercises 333

CHAPTER 10. STRUCTURES  
10.1 Finite Redundancy 342
10.2. Deformation Characteristics of a Truss Member and Presentation of a Truss Problem 343
10.3. Variational Formulations of the Truss Problem 346
10.4. The Force Method Applied to the Truss Problem 348
10.5. A Simple Example of a Truss Structure 351
10.6. Deformation Characteristics of a Frame Member 353
10.7. The Force Method Applied to a Frame Problem 355
  Exercises 369

CHAPTER 11. THE DEFORMATION THEORY OF PLASTICITY 376
11.1. The Deformation Theory of Plasticity 376
11.2. Strain-hardening Material 378
11.3. Perfectly Plastic Material 380
11.4. A Special Case of Hencky Material 383

CHAPTER 12. THE FLOW THEORY OF PLASTICITY 386
12.1. The Flow Theory of Plasticity 386
12.2. Strain-hardening Material 388
12.3. Perfectly Plastic Material 391
12.4. The Prandtl–Reuss Equation 393
12.5. The Saint-Venant–Levy–Mises Equations 395
12.6. Limit Analysis 399
12.7. Some Remarks 402

Part B—Variational Principles as a Basis of the Finite Element Method

INTRODUCTION 407

CHAPTER 13. CONVENTIONAL AND MODIFIED VARIATIONAL PRINCIPLES IN THE SMALL DISPLACEMENT THEORY OF ELASTOSTATICS 411
13.1. Presentation of a Problem 411
13.2. Conventional Variational Principles 413
13.3. Finite Elements 417
### CONTENTS

13.4. Modified Variational Principles 421  
13.5. Modified Variational Principles—continued 427  
   Exercises 430

**CHAPTER 14. CONVENTIONAL AND MODIFIED VARIATIONAL PRINCIPLES**  
**IN THE FINITE DISPLACEMENT THEORY OF ELASTOSTATICS** 432  
14.1. Presentation of a Problem 432  
14.2. Conventional Variational Principles 434  
14.3. Derivation of Modified Variational Principles from the Principle of Stationary Potential Energy 436  
14.4. A Formulation for the Compatible Model and the Modified Incremental Stiffness Method 440  
14.5. Note on the Principle of Stationary Complementary Energy in Nonlinear Elasticity 442

**CHAPTER 15. CONVENTIONAL AND MODIFIED VARIATIONAL PRINCIPLES**  
**IN ELASTODYNAMICS** 446  
15.1. Presentation of a Problem in the Small Displacement Theory of Elasto-dynamics 446  
15.2. Conventional Variational Principles in the Small Displacement Theory of Elastodynamics 447  
15.3. Gurtin’s Principle 453

**CHAPTER 16. TWO INCREMENTAL THEORIES FOR A SOLID-BODY PROBLEM**  
**WITH GEOMETRICAL AND MATERIAL NONLINEARITIES** 456  
16.1. Introduction 456  
16.2. Definitions of Strains 457  
16.3. Definitions of Stresses 459  
16.4. Relations between Stress and Strain Increments 466  
16.5. Total Lagrangian Approach 467  
16.6. Updated Lagrangian Approach 470  
   Exercises 473

**CHAPTER 17. VARIATIONAL PRINCIPLES FOR THE BENDING OF ELASTIC PLATES** 475  
17.1. Conventional Variational Principles in the Small Displacement Theory under Kirchhoff Hypothesis 475  
17.2. Modified Variational Principles in the Small Displacement Theory under Kirchhoff Hypothesis 484  
17.3. Herrmann’s Principle 489  
17.4. Variational Principles for the Large Deflection of a Plate in Stretching and Bending under the Kirchhoff Hypothesis 490  
17.5. Variational Principles for Marguerre’s Theory of a Thin Shallow Shell 494  
17.6. Conventional Variational Principles for the Bending of a Thin Plate including the Effect of Transverse Shear Deformation 497  
17.7. Some Recent Variational Treatments of Plate Bending Problems* 499  
   Exercises 508

**CHAPTER 18. NOTES ON DISCRETE ANALYSIS** 510  
18.1. Introduction 510  
18.2. Method of Weighted Residuals 511  
18.3. Rayleigh–Ritz Method 515  
18.4. Some Remarks on Discrete Analysis 517  
18.5. Boundary Element Method 518  
18.6. An Application of BEM to a Nonlinear Free Surface Fluid Problem 522  
18.7. Some Remarks on the BEM 528  
   Exercises 539
APPENDIX A.  Notes on the Stationary Property of a Function 549
APPENDIX B.  Variational Principles in Dynamics of a System of Particles 556
APPENDIX C.  Notes on the Principle of Virtual Work 568
APPENDIX D.  Notes on the Strain Energy Function and Complementary Energy Function 572
APPENDIX E.  Notes on Several Kinds of Stress Tensors in Finite Displacement Theory 576
APPENDIX F.  Vectors and Tensors 584
APPENDIX G.  Coupling of Bending and Torsion of a Beam 587
APPENDIX H.  A Beam Theory including the Effect of Transverse Shear Deformation 595
APPENDIX I.  Stress–strain Relations for a Thin Plate 599
APPENDIX J.  A Theory of Plate Bending including the Effect of Transverse Shear Deformation 601
APPENDIX K.  Specializations to Several Kinds of Shells 604
APPENDIX L.  A Note on the Haar–Karmán Principle 608
APPENDIX M.  Notes on Variational Principles for Quasi-static Problems and in the Theory of Creep 610
APPENDIX N.  Notes on the Boundary Element Method 616

INDEX 627