# TABLE OF CONTENTS

**Preface**  
xi

**Chapter 1. Some Equations of Classical Mechanics and Their Hamiltonian Properties**  
1

§1. Classical Equations of Motion of a Three-Dimensional Rigid Body  
1.1. The Euler–Poisson Equations Describing the Motion of a Heavy Rigid Body around a Fixed Point  
1.2. Integrable Euler, Lagrange, and Kovalevskaya Cases  
1.3. General Equations of Motion of a Three-Dimensional Rigid Body  

§2. Symplectic Manifolds  
2.1. Symplectic Structure in a Tangent Space to a Manifold  
2.2. Symplectic Structure on a Manifold  
2.3. Hamiltonian and Locally Hamiltonian Vector Fields and the Poisson Bracket  
2.4. Integrals of Hamiltonian Fields  
2.5. The Liouville Theorem  

§3. Hamiltonian Properties of the Equations of Motion of a Three-Dimensional Rigid Body  
34

§4. Some Information on Lie Groups and Lie Algebras Necessary for Hamiltonian Geometry  
4.1. Adjoint and Coadjoint Representations, Semisimplicity, the System of Roots and Simple Roots, Orbits, and the Canonical Symplectic Structure  
4.2. Model Example: $SL(n, \mathbb{C})$ and $sl(n, \mathbb{C})$  
4.3. Real, Compact, and Normal Subalgebras  

**Chapter 2. The Theory of Surgery on Completely Integrable Hamiltonian Systems of Differential Equations**  
55

1.1. Formulation of the Results in Four Dimensions  
55
1.2. A Short List of the Basic Data from the Classical Morse Theory 68
1.3. Topological Surgery on Liouville Tori of an Integrable Hamiltonian System upon Varying Values of a Second Integral 70
1.4. Separatrix Diagrams Cut out Nontrivial Cycles on Nonsingular Liouville Tori 73
1.5. The Topology of Hamiltonian-Level Surfaces of an Integrable System and of the Corresponding One-Dimensional Graphs 78
1.6. Proof of the Principal Classification Theorem 2.1.2 91
1.7. Proof of Claim 2.1.1 91
1.8. Proof of Theorem 2.1.1. Lower Estimates on the Number of Stable Periodic Solutions of a System 92
1.9. Proof of Corollary 2.1.5 97
1.10 Topological Obstacles for Smooth Integrability and Graphlike Manifolds. Not each Three-Dimensional Manifold Can be Realised as a Constant-Energy Manifold of an Integrable System 98
1.11. Proof of Claim 2.1.4 99

2.1. Bifurcation Diagram of the Momentum Mapping for an Integrable System. The Surgery of General Position 103
2.2. The Classification Theorem for Liouville Torus Surgery 109
2.3. Toric Handles. A Separatrix Diagram is Always Glued to a Nonsingular Liouville Torus $T^n$ Along a Nontrivial $(n - 1)$-Dimensional Cycle $T^{n-1}$ 111
2.4. Any Composition of Elementary Bifurcations (of Three Types) of Liouville Tori Is Realised for a Certain Integrable System on an Appropriate Symplectic Manifold 116
2.5. Classification of Nonintegrable Critical Submanifolds of Bott Integrals 123

§3. The Properties of Decomposition of Constant-Energy Surfaces of Integrable Systems into the Sum of Simplest Manifolds 126
3.1. A Fundamental Decomposition $Q = mI + pII + qIII + sIV + rV$ and the Structure of Singular Fibres 126
3.2. Homological Properties of Constant-Energy Surfaces 129

Chapter 3. Some General Principles of Integration of Hamiltonian Systems of Differential Equations 143
§1. Noncommutative Integration Method 143
1.1. Maximal Linear Commutative Subalgebras in the Algebra of Functions on Symplectic Manifolds 143
1.2. A Hamiltonian System Is Integrable if Its Hamiltonian is Included in a Sufficiently Large Lie Algebra of Functions 146
Table of Contents

1.3. Proof of the Theorem 149

§2. The General Properties of Invariant Submanifolds of Hamiltonian Systems 157
   2.1. Reduction of a System on One Isolated Level Surface 157
   2.2. Further Generalizations of the Noncommutative Integration Method 160

§3. Systems Completely Integrable in the Noncommutative Sense Are Often Completely Liouville-Integrable in the Conventional Sense 165
   3.1. The Formulation of the General Equivalence Hypothesis and its Validity for Compact Manifolds 165
   3.2. The Properties of Momentum Mapping of a System Integrable in the Noncommutative Sense 167
   3.3. Theorem on the Existence of Maximal Linear Commutative Algebras of Functions on Orbits in Semisimple and Reductive Lie Algebras 171
   3.4. Proof of the Hypothesis for the Case of Compact Manifolds 173
   3.5. Momentum Mapping of Systems Integrable in the Noncommutative Sense by Means of an Excessive Set of Integrals 173

§4. Liouville Integrability on Complex Symplectic Manifolds 178
   4.1. Different Notions of Complex Integrability and Their Interrelation 178
   4.2. Integrability on Complex Tori 181
   4.3. Integrability on K3-Type Surfaces 182
   4.4. Integrability on Beauville Manifolds 184
   4.5. Symplectic Structures Integrated without Degeneracies 186

Chapter 4. Integration of Concrete Hamiltonian Systems in Geometry and Mechanics. Methods and Applications 187

§1. Lie Algebras and Mechanics 187
   1.1. Embeddings of Dynamic Systems into Lie Algebras 187
   1.2. List of the Discovered Maximal Linear Commutative Algebras of Polynomials on the Orbits of Coadjoint Representations of Lie Groups 189

§2. Integrable Multidimensional Analogues of Mechanical Systems Whose Quadratic Hamiltonians are Contained in the Discovered Maximal Linear Commutative Algebras of Polynomials on Orbits of Lie Algebras 207
   2.1. The Description of Integrable Quadratic Hamiltonians 207
   2.2. Cases of Complete Integrability of Equations of Various Motions of a Rigid Body 210
   2.3. Geometric Properties of Rigid-Body Invariant Metrics on Homogeneous Spaces 216
Table of Contents

§3. Euler Equations on the Lie Algebra \( \mathfrak{so}(4) \) 220

§4. Duplication of Integrable Analogues of the Euler Equations by Means of Associative Algebra with Poincaré Duality 231
  4.1. Algorithm for Constructing Integrable Lie Algebras 231
  4.2. Frobenius Algebras and Extensions of Lie Algebras 236
  4.3. Maximal Linear Commutative Algebras of Functions on Contractions of Lie Algebras 243

§5. The Orbit Method in Hamiltonian Mechanics and Spin Dynamics of Superfluid Helium-3 250

Chapter 5. Nonintegrability of Certain Classical Hamiltonian Systems 256

§1. The Proof of Nonintegrability by the Poincaré Method 256
  1.1. Perturbation Theory and the Study of Systems Close to Integrable 256
  1.2. Nonintegrability of the Equations of Motion of a Dynamically Nonsymmetric Rigid Body with a Fixed Point 260
  1.3. Separatrix Splitting 261
  1.4. Nonintegrability in the General Case of the Kirchhoff Equations of Motion of a Rigid Body in an Ideal Liquid 266

§2. Topological Obstacles for Complete Integrability 267
  2.1. Nonintegrability of the Equations of Motion of Natural Mechanical Systems with Two Degrees of Freedom on High-Genus Surfaces 267
  2.2. Nonintegrability of Geodesic Flows on High-Genus Riemann Surfaces with Convex Boundary 272
  2.3. Nonintegrability of the Problem of \( n \) Gravitating Centres for \( n > 2 \) 275
  2.4. Nonintegrability of Several Gyroscopic Systems 277

§3. Topological Obstacles for Analytic Integrability of Geodesic Flows on Non-Simply-Connected Manifolds 281

§4. Integrability and Nonintegrability of Geodesic Flows on Two-Dimensional Surfaces, Spheres, and Tori 287
  4.1. The Holomorphic 1-Form of the Integral of a Geodesic Flow Polynomial in Momenta and the Theorem on Nonintegrability of Geodesic Flows on Compact Surfaces of Genus \( g > 1 \) in the Class of Functions Analytic in Momenta 287
  4.2. The Case of a Sphere and a Torus 291
  4.3. The Properties of Integrable Geodesic Flows on the Sphere 294

§1. Construction of the Topological Invariant 300

§2. Calculation of Topological Invariants of Certain Classical Mechanical Systems 311

§3. Morse-Type Theory for Hamiltonian Systems Integrated by Means of Non-Bott Integrals 324

References 326

Subject Index 341